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The geometry of gauge fields

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Abstract. *The aim of this work is to give a self-contained development of a differential geometric formulation of gauge theories and their interactions with the theories of fundamental particles and in particular, of the theory of Yang-Mills and Yang-Mills-Higgs fields. We discuss in detail principal and associated bundles and develop the theory of connections in a principal fiber bundle and the theory of characteristic classes. These are applied to give a general formulation of gauge theories. The special cases of the theory of Yang-Mills fields and the theory of instantons and their moduli spaces are discussed separately.*

INTRODUCTION

It is well known that physical theories use the language of mathematics for their formulation. However, the original formulation of a physical law often does not reveal its appropriate mathematical setting. Indeed the relevant mathematical setting may not even exist when the physical law is first formulated. The most well known example of this is Maxwell's equations which were formulated well before the formulation of Minkowski space and the theory of special relativity. Classical differential geometry played a fun-

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damental role in Einstein's general theory of relativity and the search for a unified theory of electromagnetism and gravitation led to continued interest in geometrical methods for some time.

However, the communication between physicists and mathematicians has been rather sporadic. Indeed, they have sometimes developed essentially the same ideas without being aware of each other's work. A recent example of this missed opportunity (see [DY1] for other examples) for communication is the development of Yang-Mills theory in physics and the theory of connections in a fiber bundle in mathematics. Trying to understand the precise relationship between these theories has led to a great deal of research by mathematicians and physicists. The problems posed and the methods of solution used in each have led to significant contributions towards a better understanding of the problems and the methods in the other. For example, the solution of the positive mass conjecture in gravitation was obtained as a result of work by Schoen and Yau [SC2] in the solution of the Calabi conjecture in differential geometry. The Calabi-Yau manifolds which arise in this investigation have also been found useful as models in superstring compactification in string theory. A complete solution for a class of Yang-Mills instantons was obtained by using methods from differential and algebraic geometry [bAT1], [AT6], [DR1]. Donaldson found a surprising application of gauge theory to the study of topology of four dimensional manifolds. The first announcement of his results [DO3] stunned the mathematical community. When combined with the work of Freedman [FR1], [FR2] one of its implications: the existence of exotic $\mathbb{R}^4$'s, was a surprising enough piece of mathematics to get into the New York Times. Since then he has found many surprising applications of his work and has developed a whole area of mathematics which may be called gauge-theoretic mathematics. Scientists often wonder about the "unreasonable effectiveness" of mathematics in the natural sciences [WI1]. It now seems that mathematicians have received an unreasonably effective (even mysterious) gift of gauge theory from physics.

Associated to the Yang-Mills equations by the adjoint representation of the gauge group on its Lie algebra are the Yang-Mills-Higgs equations. If the gauge group is non-abelian then the Yang-Mills-Higgs equations admit smooth, static solutions with finite action. These equations with the gauge group $G_{ew} = U(1) \times SU(2)$ play a fundamental role in the unified theory of electromagnetic and weak interactions (also called the electro-weak theory) developed in major part by Glashow [GL1], Salam [SA1] and Weinberg [WE1]. The subgroup of G_{ew} corresponding to $U(1)$ gives rise to the electromagnetic field while the force of weak interaction corresponds to the $SU(2)$ subgroup of G_{ew}. The electro-weak theory predicted the existence of massive vector particles (the intermediate bosons W^+, W^- and Z) corresponding to the various components of the gauge potential, which mediate the weak interactions at short distances. The experimental verification of these predictions was an important factor in the renewed interest in gauge theories as providing a suitable model for the unification of fundamental forces of nature. Since then several grand unified theories have been proposed to unify the electromagnetic, weak, and strong interactions by adjoining the group $SU(3)$ of quantum chromodynamics to the gauge group of the electro-weak theory, but their success has at best been limited. It seems that further progress may depend on a better understanding of the mathematical foundations of these theories.

The present paper is based in part on a course in "Differential Geometric Methods in Physics" that was given by the authors at the Dipartimento di Fisica, Universitá di Firenze during 1986. The course was attended by advanced graduate students in Physics and research workers in theoretical physics and pure and applied mathematics. Our paper is aimed at a similar general audience. The theory of gauge fields is currently a very active area of research in theoretical physics as well as in mathematics. However, the differential geometric foundations of gauge theories are now firmly established.

Our aim in this work is to give a self-contained development of a differential geometric formulation of gauge theories and, in particular, of the theory of Yang-Mills fields. We assume acquaintance with elements of the theory of differentiable manifolds, including the structures associated with manifolds such as tensor bundles and differential forms. We give a brief discussion of this mathematical background in section 1.1. We discuss in detail principal and associated bundles in chapter 2 and develop the theory of connections in chapter 3. In chapter 4 we discuss some concepts from algebraic topology that are frequently encountered in physical applications. The first four chapters lay the groundwork for applications to gauge theories, but the material contained in them is also useful for many other physical applications. Chapter 5 is devoted to a general formulation of gauge fields and their associated fields. The topology of the space of gauge connections and its application to the Gribov ambiguity is studied in section 5.2. A Lagrangian approach to coupled field equations is discussed in section 5.3. In chapter 6 the special cases of the theory of Yang-Mills and Yang-Mills-Higgs fields and the theory of instantons are discussed separately. We give an explicit construction of the moduli space $\mathcal{M}_1$ of the BPST-instantons of instanton number 1 and indicate the construction of the moduli space $\mathcal{M}_k$ of the complete $(8k-3)$-parameter family of instanton solutions over S^4 with gauge group $SU(2)$ and instanton number k. The investigation of the Riemannian geometry of these moduli spaces has begun only recently. The results obtained for the metric and curvature of $\mathcal{M}_1$ are given in section 6.4. In chapter 7 we give a brief discussion of the most extensively studied coupled system, namely, the system of Yang-Mills-Higgs fields and touch upon some related areas of active current research. After a brief discussion of various couplings in section 7.1 we introduce the idea of dimensional reduction in section 7.2 to study the relation of the Yang-Mills field on an $(m+1)$-dimensional manifold and its reduction to the Yang-Mills-Higgs system on an m-dimensional manifold. In section 7.3 we give some results on the monopole solutions of the Yang-Mills-Higgs system and in particular, of the Bogomolnyi equations. In the concluding section 7.4 we comment on the problem of quantization of gauge and associated fields.

Remark on references and notation

We have divided the references into two parts: books and articles. In addition to the standard texts and monographs we have also included some books which give an elementary introductory treatment of some topics. On the other hand there are also books which represent collections of papers dealing with recent research. We have included an extensive list of original research papers and review articles which have contributed to our understanding of the geometry of gauge fields.

Until recently gauge theories and the theory of connections were developed independently by physicists and mathematicians and as such there is no standard notation. We have used the notation that is primarily used in the mathematical literature but we have also taken into account the terminology that is most frequently used in physics. To help the reader we have included at the end of the paper a dictionary of terminology prepared along the lines of A. Trautman [bTR1], and [WU1], [WU2].

1. MATHEMATICAL AND PHYSICAL BACKGROUND

1.1. Mathematical background

The mathematical background required for the study of gauge theories is rather extensive and may be divided into the following parts : elements of differential geometry, fiber bundles and connections, algebraic topology of a manifold. The first two of these parts are nowadays fairly standard background for research workers in mathematical physics. Therefore in this section and in chapters 2 and 3 we give only a summary of some results from differential geometry to establish notations and to make the paper essentially self-contained. The last topic will be discussed in detail in chapter 4. There are several standard references available for material on differential geometry, see, for example, W. Greub, S. Halperin, R. Vanstone [bGR1], [bGR2], [bGR3], S. Kobayashi, K. Nomizu [bKO1], [bKO2], S. Lang [bLA1], M. Spivak [bSP2]. Some basic references for topology and related geometry and analysis are B. Booss, D. Bleecker [bBO1], J. Dugundji [bDU1], R. Palais [bPA1], I. Porteous [bPO1]. For references which also discuss some physical applications see R. Abraham, J. Marsden [bAB1], R. Abraham, J. Marsden, T. Ratiu [bAB2], Y. Choquet-Bruhat, C. DeWitt-Morette [bCH3], W. Curtis, F. Miller [bCU1], B. Felsager [bFE1], J. Marsden [bMA1], R. Sachs, H. Wu [bSA1], A. Trautman [bTR1].

The basic objects of study in differential geometry are manifolds and maps between manifolds. Roughly speaking a manifold is a topological space obtained by patching together open sets in a Banach space. For most applications this space may be taken finite-dimensional.

DEFINITION 1.1. *Let M be a connected topological space and F a Banach space. A* chart (U, ϕ) *is a pair consisting of an open set* $U \subset M$ *and a homeomorphism*

$$\phi : U \to \phi(U) \subset F,$$

where $\phi(U)$ *is an open subset of* F. M *is a* topological manifold *if* M *admits a family* $\{(U_i, \phi_i)\}_{i \in I}$ *of charts such that* $\{U_i\}_{i \in I}$ *covers* M. *This family of charts is said to be an* atlas *for* M. *If* $(U_i, \phi_i), (U_j, \phi_j)$ *are two charts and* $U_{ij} := U_i \cap U_j \neq \emptyset$ *then*

$$\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_{ij}) \to \phi_i(U_{ij})$$

is a homeomorphism. The maps ϕ_{ij} *are called* transition functions. *Various smoothness requirements are obtained by using the transition functions. For example if the* ϕ_{ij} *are* C^p*-diffeomorphisms (i.e.* ϕ_{ij} *and* ϕ_{ij}^{-1} *are of class* C^p*),* $0 < p \leq +\infty$ *,then* M *is called a* differential manifold *of class* C^p. *By definition the* dimension *of* M, $dim\ M$, *is the dimension of* F. *If* F *is* $\mathbb{R}^m$ *then* M *is called a real manifold of dimension* m. *If* F *is* $\mathbb{C}^m$ *and the transition functions are holomorphic (complex analytic) then* M *is called a complex manifold of complex dimension* m.

Complex manifolds and their physical applications are not considered in this paper. An excellent introduction to this area is given in R.O. Wells, Jr. [bWE1].

The charts allow us to give intrinsic formulations of various structures associated with manifolds.

DEFINITION 1.2. *Let* M *and* N *be differential manifolds and* $f : M \to N$. *We say that* f *is* differentiable *(or* smooth*) if, for each couple* $(U, \phi), (V, \psi)$ *of charts, of* M *and* N *respectively, such that* $f(U) \subset V$, *the* representative $\psi \circ f \circ \phi^{-1}$ *of* f *in these charts is differentiable (or smooth). The set of all smooth functions from* M *to* N *is denoted by* $\mathcal{F}(M, N)$. *When* $N = \mathbb{R}$ *we write* $\mathcal{F}(M)$ *instead of* $\mathcal{F}(M, \mathbb{R})$. *A bijective differentiable* f *is called a* diffeomorphism *if* f^{-1} *is differentiable. The set of all diffeomorphisms of* M *with itself under composition is a group denoted by* Diff(M). *Diffeomorphism is an equivalence relation.*

We observe that the same topological manifold may be given inequivalent differentiable structures. Let M be a differential manifold and $(U, \phi), (V, \psi)$ be two charts of M at $p \in M$. The triples $(\phi, p, u), (\psi, p, v), u, v \in F$, are said to be equivalent if

$$D(\psi \circ \phi^{-1})(\phi(p)) \cdot u = v$$

where D is the derivative operator in a Banach space. This is an equivalence relation between such triples. A *tangent vector* to M at p may be defined as an equivalence class $[\phi, p, u]$ of such triples. Given a smooth curve on M passing through p, we can associate with it a tangent vector to M at p. The set of tangent vectors at p is denoted by T_pM and is a vector space isomorphic to F. The set

$$TM = \bigcup_{p \in M} T_pM$$

can be given the structure of a manifold and is called the *tangent space* to M. A tangent vector $[\phi, p, u]$ at p may be identified with the directional derivative u_p^ϕ, also denoted by u_p, defined by

$$u_p : \mathcal{F}(U) \to \mathbf{R}$$

such that

$$u_p(f) = D(f \circ \phi^{-1})(\phi(p)) \cdot u.$$

If $dim\ M = m$ and $\phi : q \mapsto (x^1, x^2, \ldots, x^m)$ is a chart at p, then the tangent vectors to the coordinate curves at p are denoted by $\partial/\partial x^i$ or $\partial_i, i = 1, \ldots, m$. A smooth map

$$X : M \to TM$$

is called a *vector field* on M if $X(p) \in T_pM, \forall p \in M$. The set of all vector fields on M is denoted by $\mathcal{X}(M)$. $X \in \mathcal{X}(M)$ defines the linear map $\mathcal{F}(M) \to \mathcal{F}(M)$ by $f \mapsto Xf$ where $(Xf)(p) = X(p)f$. If $X, Y \in \mathcal{X}(M)$ then the commutator $[X, Y] = X \circ Y - Y \circ X$ is in $\mathcal{X}(M)$. If $f \in \mathcal{F}(M, N)$, the *tangent of* f *at* p, denoted by T_pf or $f_*(p)$, is the map

$$T_pf : T_pM \to T_{f(p)}N$$

such that

$$T_pf(u_p) \cdot g = u_p(g \circ f).$$

The *tangent of* f, denoted by Tf or f_*, is the map of TM to TN whose restriction to T_pM is T_pf.

By replacing T_pM with various tensor spaces on T_pM, a construction similar to the above for the tangent space, defines the tensor spaces on M. We denote by T_s^rM the *tensor space of type* (r, s) of contravariant degree r and covariant degree s, i.e.

$$T_s^rM = \bigcup_{p \in M} T_s^r(T_pM)$$

where

$$T_s^r(T_pM) = \underbrace{T_pM \otimes \ldots \otimes T_pM}_{r \text{ times}} \otimes \underbrace{(T_pM)^* \otimes \ldots \otimes (T_pM)^*}_{s \text{ times}}.$$

A smooth map

$$t : M \to T_s^r M$$

is called a *tensor field of type* (r,s) on M if $t(p) \in T_s^r(T_pM), \forall p \in M$. We note that, if M is finite dimensional, then $T_s^r(T_pM)$ may be identified with a space of multilinear maps as follows. The element

$$u_1 \otimes \ldots \otimes u_r \otimes \alpha^1 \otimes \ldots \otimes \alpha^s \in T_s^r(T_pM)$$

is identified with the map of $T_s^r((T_pM)^*)$ to $\mathbf{R}$ defined by

$$(\beta^1, \ldots, \beta^r, v_1, \ldots, v_s) \mapsto \beta^1(u_1) \ldots \beta^r(u_r)\alpha^1(v_1) \ldots \alpha^s(v_s).$$

This map is extended by linearity to all of $T_s^r(T_pM)$. We note that, if $dim\ M = m$, then dim $T_s^r M = m^{r+s}$. We observe that $T_0^1 M = TM$. The space $T_1^0 M$ is denoted by T^*M and is called the *cotangent space* of M. We define $T_0^0 M := \mathcal{F}(M)$.

Let $g \in T_2^0 M$; we say that g is *non-degenerate* if, for each $p \in M$, $g(p)$ is non-degenerate, i.e.

$$g(p)(u,v) = 0, \quad \forall v \in T_pM \Rightarrow u = 0.$$

A *pseudo-metric* on M is a $g \in T_2^0 M$ which is symmetric and non-degenerate. Each $g(p)$ then defines an inner product on T_pM of signature (r,s) and index $i_g = s$, where $r + s =$ dim M. If g is a pseudo-metric on M of index s then we say that (M,g) is a *pseudo-Riemannian* manifold of index s. If $s = 0$, i.e. $\forall p \in M,\ \ g(p)$ is positive definite, then we say that (M,g) is a *Riemannian* manifold. If $s = 1$, i.e. $\forall p \in M$, the signature of $g(p)$ is (dim $M - 1, 1)$ then we say that (M,g) is a *Lorentz* manifold. We note that a pseudo-metric g induces an inner product on all tensor spaces, which we also denote by g. There are several important differences in both the local and global properties between Riemannian and pseudo-Riemannian geometry (see, for example, J.K. Beem, P.E. Ehrlich [bBE1], B. O'Neill [bON1]). Until recently, most physical applications involved pseudo-Riemannian (in particular, Lorentz) manifolds. However, the discovery of instantons and their possible role in quantum field theory and subsequent development of the so-called Euclidean gauge theories has led to extensive use of Riemannian geometry in physical applications.

We define $A^0 M := \mathcal{F}(M)$ and denote by $A^k M$, $k \geq 1$, the manifold

$$A^k M = \bigcup_{p \in M} A^k(T_p M)$$

where

$$A^k(T_p M) = \underbrace{T_p^* M \wedge \ldots \wedge T_p^* M}_{k \text{ times}}$$

is the vector space of exterior k-forms on $T_p^* M$. We note that $A^1 M = T^* M$. If dim $M = m$ and $\{e_1, \ldots, e_m\}$ is a basis for $T_p^* M$ then

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{1 \leq i_1 < \ldots < i_k \leq m}$$

is a basis for $A^k(T_p M)$. Thus

$$\dim A^k M = \binom{m}{k}$$

A smooth map

$$\alpha : M \to A^k M$$

is called a *k-form* on M if $\alpha(p) \in A^k(T_p^* M)$, $\forall p \in M$. The space of k-forms on M is denoted by $\Lambda^k(M)$. We define the (graded) *exterior algebra* on M, $\Lambda(M)$, by

$$\Lambda(M) = \bigoplus_{k=0}^{+\infty} \Lambda^k(M).$$

If dim $M = m$, $\Lambda^k(M) = \{0\}$ for $k > m$. An m-form ν is called a *volume form* or simply *volume* on M if $\forall p \in M$, $\nu(p) \neq 0$. M is said to be *orientable* if it admits a volume. Two volumes ν, ω on M are said to be *equivalent* if $\omega = f\nu$ for some $f \in \mathcal{F}(M)$ such that $f(m) > 0$, $\forall m \in M$. An *orientation* of an orientable manifold M is an equivalence class $[\nu]$ of volumes on M. If (M, g) is an oriented, pseudo-Riemannian manifold with orientation $[\nu]$, then we define the *metric volume form* μ by $\mu = \nu / |g(\nu,\nu)|^{1/2}$. On an oriented pseudo-Riemannian manifold (M, g) with metric volume μ, we define the *Hodge star operator*

$$* : \Lambda(M) \to \Lambda(M)$$

as follows. For $\beta \in \Lambda^k(M)$, $*\beta \in \Lambda^{m-k}(M)$ is the unique form such that

$$\alpha \wedge *\beta = g(\alpha, \beta)\mu, \quad \forall \alpha \in \Lambda^k(M).$$

Let $f \in \mathcal{F}(M,N)$; then f induces the following map

$$f^* : \Lambda(N) \to \Lambda(M),$$

called the *pull-back* map, defined as follows. If $\alpha \in \Lambda^0(N) = \mathcal{F}(N)$ then $f^*\alpha = \alpha \circ f \in \Lambda^0(M) = \mathcal{F}(M)$. If $\alpha \in \Lambda^k(N)$, $k \geq 1$, then $f^*\alpha \in \Lambda^k(M)$ is defined by

$$(f^*\alpha)(p)(u_1, \ldots, u_k) =$$
$$= \alpha(f(p))(T_pf(u_1), \ldots, T_pf(u_k)), \ \forall u_1, \ldots, u_k \in T_pM$$

If $f : M \to N$ is a diffeomorphism and $X \in \mathcal{X}(N)$, the element f^*X of $\mathcal{X}(M)$ defined by

$$f^*X = Tf^{-1} \circ X \circ f$$

is called the *pull-back* of X by f . Then one can extend the definition of the pull-back map f^* (when f is a diffeomorphism) to the tensor fields of type (r, s) . In particular if $X_1, \ldots, X_r \in \mathcal{X}(N)$ and $\alpha^1, \ldots, \alpha^s \in \Lambda^1(N)$, we have

$$f^*(X_1 \otimes \ldots X_r \otimes \alpha^1 \otimes \ldots \alpha^s) =$$
$$= (f^*X_1) \otimes \ldots \otimes (f^*X_r) \otimes (f^*\alpha^1) \otimes \ldots \otimes (f^*\alpha^s).$$

Given $X \in \mathcal{X}(M)$ and $p \in M$, an *integral curve* of X through p is a smooth curve

$$c : I \to M$$

where I is an open interval around $0 \in \mathbb{R}$, such that $c(0) = p$ and

$$\dot{c}(t) := Tc(t, 1) = X(c(t))$$

$\forall t \in I$. A *local flow* of X at $p \in M$ is a map

$$F : I \times U \to M,$$

where U is an open neighborhood of p, such that, $\forall q \in U$, the map $F_q : I \to M$ defined by

$$F_q : t \mapsto F(t, q)$$

is an integral curve of X through q. One can show that, $\forall X \in \mathcal{X}(M)$, $\forall p \in M$, a local flow $F : I \times U \to M$ of X at p exists and the map F_t defined by

$$F_t(q) = F(t,q), \quad \forall q \in U$$

is a diffeomorphism of U onto some open subset U_t of M.

DEFINITION 1.3. *Let $X \in \mathcal{X}(M)$, and η be a tensor field of type (r,s) on M. The* Lie derivative $L_X\eta$ *of η with respect to X is the tensor field of type (r,s) defined by*

$$(L_X\eta)(p) = \frac{d}{dt}[((F_t^*\eta)(p)]|_{t=0}$$

$\forall p \in M$, where $F : I \times U \to M$ is a local flow of X at p. The above definition also applies to differential forms; then $\eta \in \Lambda^k(M)$ implies $L_X\eta \in \Lambda^k(M)$.

It can be shown that the definition of Lie derivative given above is independent of the choice of a local flow.

DEFINITION 1.4. *The* exterior differential operator d *of degree 1 on $\Lambda(M)$ is the map $d : \Lambda(M) \to \Lambda(M)$ defined as follows. If $f \in \Lambda^0(M)$ then $df \in \Lambda^1(M)$ is defined by*

$$df \cdot X = Xf.$$

If $\omega \in \Lambda^k(M)$, $k > 0$, then $d\omega$ is the unique $(k+1)$-form such that

$$d\omega(X_0, X_1, \ldots, X_k) =$$
$$= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0, X_1, \ldots, \hat{X}_i, \ldots, X_k)) +$$
$$+ \sum_{0 \le i < j \le k}(-1)^{i+j}\omega(L_{X_i}X_j, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$$

where $\hat{X}_h$ denotes suppression of X_h.

$\alpha \in \Lambda(M)$ is *closed* (resp. *exact*) if $d\alpha = 0$ (resp. $\alpha = d\beta$). From the above definition it follows that

$$d^2 := d \circ d = 0.$$

Thus every exact form is closed. The converse of this statement is in general valid only locally, i.e. if $\alpha \in \Lambda(M)$ is closed then, $\forall p \in M$, there exists a neighborhood U of p such that $\alpha_{|U}$ is exact. This statement is called the *Poincaré lemma.*

DEFINITION 1.5. *Let* (M, g) *be an* m*-dimensional oriented pseudo-Riemannian manifold of index* i_g. *The* codifferential δ *of degree* -1 *is the map* $\delta : \Lambda(M) \to \Lambda(M)$ *which on* $\Lambda^k(M)$ *is defined by*

$$\delta := (-1)^{i_g+mk+m+1} * d*$$

where $*$ *is the Hodge star operator.*

We observe that if $f \in \Lambda^0(M) = \mathcal{F}(M)$, then $\delta f = 0$.

DEFINITION 1.6. *Let* $X \in \mathcal{X}(M)$ *and* $\alpha \in \Lambda(M)$. *The* inner multiplication $i_X\alpha$ *is defined as follows. If* $\alpha \in \Lambda^0(M) = \mathcal{F}(M)$, *then* $i_X\alpha = 0$. *If* $\alpha \in \Lambda^k(M)$, $k \geq 1$, *then* $i_X\alpha \in \Lambda^{k-1}(M)$ *is defined by*

$$i_X\alpha(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}).$$

In the following theorem we collect together some important properties of the operators L_X, d, i_X.

THEOREM 1.1. (i) *If* $f \in \mathcal{F}(M,N)$ *and* $\alpha \in \Lambda(N)$, *then* $f^*\alpha \in \Lambda(M)$ *and* $d(f^*\alpha) = f^*(d\alpha)$.

(ii) $\mathcal{X}(M)$ *is a Lie algebra with the product* $[X,Y]$ *and*

$$[X,Y] = L_XY, \quad X,Y \in \mathcal{X}(M).$$

(iii) $L_X = i_X \circ d + d \circ i_X$ *on* $\Lambda(M)$.

(iv) $dL_X = L_Xd$ *on* $\Lambda(M)$.

(v) $i_{[X,Y]} = L_Xi_Y - i_YL_X$ *on* $\Lambda(M)$. ∎

A manifold sometimes carries an additional mathematical structure. An important example of this is furnished by a Lie group.

DEFINITION 1.7. *A (finite-dimensional)* Lie group G *is a (finite-dimensional) manifold which carries a compatible group structure, i.e. the operations of multiplication and taking the inverse are smooth.*

A *Lie group (left) action* of a Lie group G on a manifold M is a smooth map

$$L : G \times M \to M$$

such that

$$L_g : M \to M$$

defined by $L_g(x) = L(g, x)$ (also denoted by gx), is a diffeomorphism of M, $\forall g \in G$ and

$$\forall g_1, g_2 \in G, \; L_{g_1 g_2} = L_{g_1} \circ L_{g_2} \quad \text{and} \quad L_e = id_M,$$

where e is the identity element of G. This may be expressed by saying that the map $L : G \to \mathrm{Diff}(M)$ such that $g \mapsto L_g$ is a group homomorphism. The *orbit* of $x \in M$ under the G-action is the subset $\{gx \mid g \in G\}$ of M, also denoted by Gx. The set of the orbits of the G-action L on M is denoted by M/L or by M/G when L is understood. A G-action on M is said to be *transitive* if there is just one orbit; in this case we also say that G acts transitively on M. If $x \in M$, then the *isotropy group* H_x of the G-action is defined by

$$H_x = \{g \in G \mid gx = x\}.$$

A G-action on M is said to be *free* if $gx = x$ for some $x \in M$, implies $g = e$, i.e. if $H_x = \{e\}$, $\forall x \in M$. A right G-action on M may be defined similarly; then, with obvious changes, one has the related notions of orbit, transitive action, etc..

G acts on itself by left multiplication. A vector field $X \in \mathcal{X}(G)$ on G is said to be *left invariant* if it is invariant under this left action L, i.e.

$$(L_g)^* X = X, \qquad \forall g \in G.$$

The set of all left invariant vector fields is a Lie subalgebra of $\mathcal{X}(G)$ and is called the Lie algebra of the group G and is denoted by $\mathbf{g}$. The tangent space T_eG to G at the identity e is isomorphic, as a vector space, to $\mathbf{g}$. This isomorphism is used to make T_eG into a Lie algebra isomorphic to $\mathbf{g}$. If E_i, $1 \leq i \leq m$, is a basis for $\mathbf{g}$ then we have

$$[E_j, E_k] = c^i_{jk} E_i,$$

where we have used the Einstein summation convention of summing over repeated indices. The constants c^i_{jk} are called the *structure constants* of $\mathbf{g}$ with respect to the basis $\{E_i\}$. They characterize the Lie algebra $\mathbf{g}$ and satisfy the following relations:

(i) $c^i_{jk} = -c^i_{kj}$,

(ii) $c^i_{jk}c^l_{im} + c^i_{km}c^l_{ij} + c^i_{mj}c^l_{ik} = 0$ (Jacobi identity).

$A \in \mathbf{g}$ generates a global one-parameter group ϕ_t of diffeomorphisms of G. We define

$$\exp(tA) := \phi_t(e).$$

Thus we have the map $\exp : \mathbf{g} \to G$ defined by:

$$\exp : A \mapsto \phi_1(e),$$

which is a homeomorphism on some neighborhood of $0 \in \mathbf{g}$. This homeomorphism can be used to define a special coordinate chart on G called the *normal coordinate chart.* The map exp is called the *exponential mapping* and coincides with the usual exponential functions for matrix groups and algebras. The *adjoint action* Ad of G on itself is defined by

$$\mathrm{Ad} : G \to \mathrm{Aut}\, G$$

such that $h \mapsto \mathrm{Ad}(h)$ where $\mathrm{Ad}(h) : G \to G$ is defined by

$$\mathrm{Ad}(h)g = hgh^{-1}.$$

This action induces an action ad of G on $\mathbf{g}$ which is a representation of G, called the *adjoint representation* of G on $\mathbf{g}$. The contragradient of this representation is called the *coadjoint representation* of G on $\mathbf{g}^*$ and is important in the theory of representations. The study of this representation is the starting point of the Kostant-Kirillov-Souriau theory of geometric quantization (see R. Abraham, J. Marsden [bAB1], [MA10], [MA11] and references therein). If H is a closed subgroup of G then H is a Lie subgroup and the quotient G/H is a differential manifold which is called a *homogeneous space* of G. In fact if G acts transitively on M (i.e. given $x, y \in M$ there exists $g \in G$ such that $y = gx$) and if H is the isotropy subgroup of a fixed point in M, then M is diffeomorphic to G/H.

EXAMPLE 1.1. *The rotation group $SO(n+1)$ of $\mathbf{R}^{n+1}$ acts on the sphere S^n transitively. The isotropy group at $(1, 0, \ldots, 0)$ may be identified with $SO(n)$. Thus the sphere is a homogeneous space of the group $SO(n+1)$*

$$S^n = SO(n+1)/SO(n).$$

The conformal group $SO(n,1)$ acts transitively on the open unit ball $B^n \subset \mathbf{R}^{n+1}$ with isotropy group at the origin $SO(n)$. Thus B^n is a homogeneous space of the conformal group, called the Poincaré model of the hyperbolic space i.e.

$$B^n = SO(n,1)/SO(n).$$

For $n = 5$ this construction occurs in the study of the moduli space of BPST instantons.

A standard reference for the theory of Lie groups is S. Helgason [bHE1].

1.2. Physical background

Maxwell's electromagnetic theory provides the simplest example of a gauge theory, with the field equations being given by Maxwell's equations. We therefore begin with a brief review of Maxwell's equations, which in classical form are given by:

$$\begin{aligned} &\text{div } B = 0 \\ &\text{curl } E = -\partial B/\partial t \\ &\text{div } E = \rho \\ &\text{curl } B = J + \partial E/\partial t \end{aligned}$$

where the electric field E and the magnetic field B are time dependent vector fields on some subset of $\mathbf{R}^3$ and ρ and J are the charge and current density respectively. These equations unified the separate theories of electricity and magnetism and paved the way for important advances in both experimental and theoretical physics. With the introduction of the four dimensional Minkowski space-time M^4 it became possible to describe both the electric and magnetic fields as part of a skew-symmetric tensor field, or a differential 2-form F, on M^4 as follows. Using the standard chart on M^4 and the induced bases on the tensor spaces, the tensor F has the components given by:

$$F_{k4} = E_k, 1 \le k \le 3 \quad F_{12} = B_3, \quad F_{23} = B_1, \quad F_{31} = B_2.$$

Maxwell's equations written in terms of F are

$$dF = 0, \quad \delta F = j,$$

where $j = (J, \rho)$ is the current density 1-form and $\delta = *d*$ is the codifferential operator on 2-forms of M^4. The source-free field equations are obtained by setting $j = 0$ and can be written as

$$dF = 0, \quad d * F = 0.$$

It is well known that the Maxwell's equations are globally invariant under the conformal group and in particular, under the Lorentz group. Their Lorentz invariance is the starting point of Einstein's theory of Special Relativity. They are also invariant under a local group of transformations known as gauge transformations. This invariance arises as follows. On M^4 the equation $dF = 0$ implies that the field F is derivable from a 4-potential or 1-form A, i.e. $F = dA$. However, the potential A is not uniquely determined. If B is another potential such that $F = dB$, then $d(B - A) = 0$ and this implies that $B - A = d\psi, \quad \psi \in \mathcal{F}(M^4)$. Thus we may think of the potential B as obtained by a gauge transformation of A by ψ. Since ψ is real valued this corresponds to a change of scale at each point. H. Weyl [WE3], [WE4] sought to incorporate the electromagnetic field into the geometric structures associated to the space-time as arising from local scale invariance. He referred to this scale invariance as «eich-invarianz» and this is the origin of the modern term gauge invariance. In fact with slight modifications replacing local scale by local phase taking values in the unitary group $U(1)$, one obtains a formulation of Maxwell's equations as gauge field equations. We will discuss this formulation in section 6.1.

Maxwell's equations written by using F admit immediate generalization to the case when the Minkowski space is replaced by an arbitrary four dimensional Lorentz manifold. According to Einstein's theory of gravity the gravitational field is described by the Lorentz metric of the space-time manifold. Thus it was natural to look for a unified theory of gravity and electromagnetism. One approach leads to the Kaluza-Klein theory [KA1], [KL1] which uses a five dimensional pseudo-Riemannian manifold and a suitable $(4+1)$-dimensional decomposition to obtain Einstein's and Maxwell's equations from the five dimensional metric. The ideas of Kaluza-Klein theory have been applied to higher dimensional manifolds to study coupled field equations and the problem of dimensional reduction. For a modern treatment of Kaluza-Klein theories, see R. Hermann [bHE2], R. Coquereaux, A. Jadczyk [bCO1].

In 1954 Yang and Mills [YA7], [YA8] obtained the following now well known gauge field equations for the vector potential b_μ of isotopic spin in interaction with a field ψ of isotopic spin $1/2$:

$$\partial f_{\mu\nu}/\partial x_\nu + 2\epsilon(b_\nu \times f_{\mu\nu}) + J_\mu = 0$$

where,

$$f_{\mu\nu} = \partial b_\mu/\partial x_\nu - \partial b_\nu/\partial x_\mu - 2\epsilon b_\mu \times b_\nu,$$

and J_μ is the current density of the source field ψ. There was no immediate physical application of these equations since they seemed to predict massless gauge particles as in Maxwell's theory. In Maxwell's theory the massless particle is identified as photon.

No such identification could be made for the massless particles predicted by Yang-Mills theory. This difficulty is overcome by the introduction of the «Higgs mechanism» [HI2] which shows how spontaneous symmetry breaking can give rise to massive gauge vector bosons by a gauge transformation to a particular gauge called the unitary gauge (see section 6.5). This paved the way for a gauge theoretic formulation of the electro-weak theory and subsequent development of the general framework for gauge theories.

The Yang-Mills equations may be thought of as a matrix valued generalization of the equations for the classical vector potential of Maxwell's theory. The gauge field that they obtained turns out to be the curvature of a connection in an $SU(2)$ principal fiber bundle. The general theory of such connections was developed in 1950 by Ehresman [EH1]. However it was not until 1975 that the identification between curvature and gauge field was made. This identification unleashed a flurry of activity among both physicists and mathematicians and has already had great successes some of which were indicated in the introduction. Since the physical and mathematical theories have developed independently each has its well established terminology. We have used the notation that is primarily used in the mathematical literature but we have also taken into account the terminology that is most frequently used in physics. To help the reader we have given at the end of the paper a dictionary indicating the correspondence between the terminologies of physics and mathematics, prepared along the lines of A. Trautman [bTR1], and T. Wu [WU1]. The physical literature on gauge theory is vast and is in general aimed at applications to elementary particle physics and quantum field theory. We cite here only a few references which may be consulted to gain some understanding of these and related aspects of gauge theories. They are T. Cheng, L. Li [bCH2], L. Fadeev, A. Slavnov [bFA1], A. Jaffe [JA6], C. Quigg [bQU1], and G. 't Hooft [TH1]. For quantum field theories and related topics, see for example, R. Feynman, A. Hibbs [bFE2], C. Schulman [bSC1], P. Goddard, P. Mansfield [GO1].

2. PRINCIPAL BUNDLES AND THEIR ASSOCIATED BUNDLES

2.1. Principal bundles

We begin by recalling the definition of a differentiable fiber bundle.

DEFINITION 2.1. *A differentiable* fiber bundle E *over* B *is a quadruple* $\zeta = (E, B, \pi, F)$, *where* E, B, F *are differentiable manifolds and the map* $\pi : E \to B$ *is an open differentiable surjection satisfying the following* local triviality property :

(LT) *There exists an open covering* $\{U_i\}_{i\in I}$ *of* B *and a family* ψ_i *of diffeomorphisms* $\psi_i : U_i \times F \to \pi^{-1}(U_i)$, $\forall i \in I$ *satisfying the conditions* $(\pi \circ \psi_i)(x, g) = x$, $\forall (x, g) \in U_i \times F$.

The family $\{(U_i, \psi_i)\}_{i\in I}$ is called a *local coordinate representation* or a *local triv-*

ialization of the bundle ζ. E is called the *total space* or the *bundle space*, B the *base space*, π the *bundle projection* of E on B, and F the *standard* or *typical fiber.* $E_x := \pi^{-1}(x)$, is called the *fiber of* E *over* $x \in B$.

The bundle $\tau = (B \times F, B, \pi, F)$, where π is the projection onto the first factor, is called the *trivial bundle* over B with fiber F. The total space of the trivial bundle τ is $B \times F$. For this reason a general fiber bundle is sometimes called a *twisted product* of B (the base) and F (the standard fiber). A fiber bundle $\zeta = (E, B, \pi, F)$ is sometimes indicated by a diagram as follows:

$$\begin{array}{c} E \\ \big\downarrow{\scriptstyle F} \\ B \end{array}$$

One can verify that the map $\psi_{i,x} : F \to E_x$ defined by $g \mapsto \psi_i(x, g)$, $\forall x \in U_i$, $\forall i \in I$, is a diffeomorphism. If $U_i \cap U_j \neq \emptyset$ then we write $U_{ij} = U_i \cap U_j$ and define

$$\text{(2.1)} \qquad \psi_{ij} : U_{ij} \to Diff(F) \qquad \text{by} \qquad \psi_{ij}(x) = \psi_{i,x}^{-1} \circ \psi_{j,x}.$$

The functions ψ_{ij} are called the *transition functions* for the local representation. They satisfy the following condition:

$$\text{(2.2)} \qquad \psi_{ij}(x) \circ \psi_{jk}(x) \circ \psi_{ki}(x) = id_F, \ \forall x \in U_{ijk},$$

where we have written $U_{ijk} = U_i \cap U_j \cap U_k$. The condition (2.2) is referred to as the *cocycle condition* on the transition functions. We note that, given the base B, the typical fiber F and a family of transition functions $\{\psi_{ij}\}$ satisfying the cocycle condition (2.2), it is possible to construct the fiber bundle E. Given the bundles $\zeta = (E, B, \pi, F), \zeta' = (E', B', \pi', F')$ a *bundle morphism* f from ζ to ζ' is a differentiable map $f : E \to E'$ such that f maps the fibers of E smoothly to the fibers of E' and therefore, induces a smooth map of B to B' denoted by f_0. Thus we have the following commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ B & \xrightarrow[f_0]{} & B' \end{array}$$

If $B = B'$, f is injective and $f_0 = id_B$, then we say that ζ is a *subbundle* of ζ'. If $\zeta = (E, B, \pi, F)$ is a fiber bundle we frequently denote by E the fiber bundle ζ when

B, π, F are understood. If $\zeta = (E, B, \pi, F)$ is a fiber bundle and $h \in \mathcal{F}(M, B)$, where M is a manifold, then we can define a bundle $h^*\zeta = (h^*E, M, h^*\pi, F)$, called the *pullback* of the bundle E to M as follows. h^*E is the subset of $M \times E$ of the couples $(p, a) \in M \times E$ such that $h(p) = \pi(a)$ and one can show that it is a closed submanifold of $M \times E$. $h^*\pi$ is the restriction to h^*E of the natural projection of $M \times E$ onto M. The map h lifts to a unique bundle map $\hat{h} : h^*E \to E$ such that the following diagram commutes:

$$\begin{array}{ccc} h^*E & \xrightarrow{\hat{h}} & E \\ \Big\downarrow{\scriptstyle h^*\pi} & & \Big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow[h]{} & B \end{array}$$

A smooth map $s : B \to E$ such that $\pi \circ s = id_B$ is called a (smooth) *section* of the fiber bundle E over B. We denote by $\Gamma(B, E)$ or simply $\Gamma(E)$ the space of sections of E over B. If $U \subset B$ is open then we denote by $\Gamma(E_{|U})$ the set of sections of the bundle E restricted to U. If $p \in U$ and $s \in \Gamma(E_{|U})$ then we say that s is a *local section* of E *at* p.

Choosing local coordinates $\{ x^i \mid 1 \leq i \leq m \}$ in a neighborhood of $p \in B$ and $\{ x^i, y^j \mid 1 \leq i \leq m, \ 1 \leq j \leq n \}$ in a neighborhood of $s(p) \in E$, we can think of s as a function from $\mathbb{R}^m$ to $\mathbb{R}^{m+n}$:

$$s : (x^i) \to (x^i, \ y^j(x^1, x^2, \cdots, x^m)).$$

The Taylor expansion of s at p clearly depends on the local coordinates chosen. However, if two local sections s and t have the same k-th order Taylor expansion at p in one coordinate system, then they have the same k-th order expansion in any other coordinate system. This observation can be used to define an equivalence relation on local sections at p. An equivalence class determined at p by the section s is called the *k-jet* of s at p and is denoted by $j^k(s)_p$. We define the *k-jet of sections* of E over B, $J^k(E/B)$ (also denoted by $J^k(E)$), by

$$J^k(E/B) := \{ j^k(s)_p \mid p \in B, \ s \text{ is a local section of } E \text{ at } p \}.$$

$J^k(E/B)$ is a fiber bundle over B with projection $\pi^k : J^k(E) \to B$ defined by $\pi^k(j^k(s)_p) = p$, and a fiber bundle over $J^l(E/B)$, $0 \leq l \leq k$, with projection $\pi^k_l : J^k(E) \to J^l(E)$ defined by $\pi^k_l(j^k(s)_p) = j^l(s)_p$. In particular $J^k(E)$ is a fiber bundle over $J^0(E) = E$. A section $s \in \Gamma(E)$ induces a section $j^k(s) \in \Gamma(J^k(E))$ defined by $j^k(s)(p) = j^k(s)_p$. We call $j^k(s)$ the *k-jet extension* of s. The map $j^k : \Gamma(E) \to \Gamma(J^k(E))$ defined by:

$$s \mapsto j^k(s)$$

is called the *k-jet extension map*. If M, N are manifolds then we define the space $J^k(M,N)$ of k-jets of maps of M to N by

$$J^k(M,N) = J^k((M \times N)/M),$$

where $M \times N$ is regarded as a trivial fiber bundle over M. It is possible to define the bundle $J^k(M,N)$ directly by considering the Taylor expansion of local maps up to order k. Jet bundles play a fundamental role in the geometrical formulation of variational problems and in particular of Lagrangian theories [GA1], [GO2], [KU1].

Let $\zeta = (E, B, \pi, F)$ be a fiber bundle. If a Lie group G is a subgroup of $Diff(F)$ such that, for each transition function ψ_{ij} of ζ, $\psi_{ij}(x) \in G$, $\forall x \in U_{ij}$ and ψ_{ij} is a smooth map of U_{ij} into G, then we say that ζ is a *fiber bundle with structure group* G. We now give the two most important cases of fiber bundles with structure group.

DEFINITION 2.2. *A fiber bundle* $\zeta = (E, B, \pi, F)$ *with structure group* G *is called a* vector bundle *with fiber type* F *if* F *is a finite dimensional Banach space and* G *is the Lie group of the linear diffeomorphisms of* F. *In particular if* F *is a real (resp. complex) vector space of dimension* n *and* $G = GL(n, \mathbb{R})$ *(resp.* $G = GL(n, \mathbb{C})$*) then we call* ζ *a* real *(resp.* complex*)* vector bundle of rank n.

We note that in the case of a vector bundle of rank n the transition functions turn out to be automatically smooth. Let E, H be two vector bundles over B. The algebraic operations on vector spaces can be extended to define vector bundles such as $E \oplus H$, $E \otimes H$, $Hom(E, H)$ by using pointwise operations on fibers over B. In particular, we can form the bundle $(A^k B) \otimes E$. The sections of this bundle are called *k-forms* on B *with values in the vector bundle* E or simply *vector bundle valued* (E-valued) k-forms. We write $\Lambda^k(B, E)$ for the space of sections $\Gamma((A^k B) \otimes E)$. Thus $\alpha \in \Lambda^k(B, E)$ can be regarded as defining for each $x \in B$ a k-linear, anti-symmetric map α_x of $T_x B$ into E_x. In particular, $\Lambda^0(B, E) = \Gamma(E)$. If E is a trivial vector bundle with fiber V, then we call $\Lambda^k(B, E)$ the space of *k-forms with values in the vector space* V or *vector valued* (V-valued) k-forms and denote it by $\Lambda^k(B, V)$.

Let V_1, V_2, V_3 be vector spaces and $h : V_1 \times V_2 \to V_3$ a bilinear form. Let $\alpha \in \Lambda^p(B, V_1), \beta \in \Lambda^q(B, V_2)$; then we define $\alpha \wedge_h \beta \in \Lambda^{p+q}(B, V_3)$ as follows. Let $\{u_i\}$ be a basis of V_1 and $\{v_j\}$ a basis for V_2. Then

$$\alpha = \alpha^i u_i, \quad \beta = \beta^j v_j,$$

where $\alpha^i \in \Lambda^p(B)$ and $\beta^j \in \Lambda^q(B)$. Then

$$\alpha \wedge_h \beta := \alpha^i \wedge \beta^j h(u_i, v_j).$$

There are several important special cases of this operation. For example if h is an inner product on a vector space V, then $\alpha \wedge_h \beta \in \Lambda^{p+q}(B)$. If V is a Lie algebra and h is the Lie bracket, then it is customary to denote $\alpha \wedge_h \beta$ by $[\alpha, \beta]$. Thus

$$[\alpha, \beta] := \alpha^i \wedge \beta^j [u_i, u_j] = c_{ij}^k \alpha^i \wedge \beta^j u_k,$$

where c_{ij}^k are the structure constants of the Lie algebra. If B is a pseudo-Riemannian manifold with metric g and h is an inner product on V then $\Lambda^k(B, V)$ becomes an inner product space, with inner product denoted by $\langle\ ,\ \rangle_{(g,h)}$, defined by

$$\langle \alpha, \beta \rangle_{(g,h)} = \langle \alpha^i, \beta^j \rangle_g h(u_i, u_j),$$

where $\langle\ ,\ \rangle_g$ is the inner product on $\Lambda^k(B)$ induced by g. In particular if V is a semisimple Lie algebra, then a multiple of the Killing form is a positive definite inner product on V. The inner product $\langle\ ,\ \rangle_{(g,h)}$, with h the Killing form and g a Riemannian metric, is positive definite and can be used to define the norm or energy of α by

$$||\alpha|| := \sqrt{\langle \alpha, \alpha \rangle_{(g,h)}}.$$

The constructions discussed above for vector valued forms can be extended to apply to vector bundle valued forms by using their pointwise vector space structures.

EXAMPLE 2.1. *Let TM be the tangent space of the m-dimensional manifold M and $\pi : TM \to M$ the canonical projection. Then $(TM, M, \pi, \mathbf{R}^m)$ is a vector bundle of rank m, called the* tangent bundle *of M. A* k-dimensional distribution *on M is a vector bundle of rank k over M which is a subbundle of the tangent bundle of M.*

DEFINITION 2.3. *A fiber bundle $\zeta = (P, M, \pi, F)$ with structure group G is called a* principal fiber bundle *over M with structure group G if F is a Lie group and G is the Lie group of the diffeomorphisms h of F such that*

$$h(g_1 g_2) = h(g_1) g_2.$$

We note that G is isomorphic to F. A principal bundle over M with structure group G is denoted by $P(M, G)$.

Equivalently a principal bundle $P(M, G)$ with structure group G over M may be defined as follows.

DEFINITION 2.4. *A principal bundle* $P(M,G)$ *with structure group* G *over* M *is a fiber bundle* (P,M,π,G) *with a free right action* ρ *of* G *on* P *such that*

(i) *the orbits of* ρ *are the fibers of* $\pi : P \to M$ *i.e.* π *may be identified with the canonical projection* $P \to P/G$;

(ii) $\forall \psi : U \times G \to \pi^{-1}(U)$ *which is a local trivialization of* P, *writing* $u_x s$ *in the place of* $\rho(u_x, s)$, *one has*

$$\psi_x^{-1}(u_x s) = \psi_x^{-1}(u_x)s, \quad \forall u_x \in P_x,\ s \in G.$$

We give below an example of a principal bundle that is naturally associated with every manifold.

EXAMPLE 2.2. (Bundle of frames) *Let* M *be an* m*-dimensional manifold. A* frame $u = (u_1, \cdots, u_m)$ *at a point* $x \in M$ *is an ordered basis of the tangent space* $T_x M$. *Let*

$$L_x(M) = \{u \mid u \text{ is a frame at } x \in M\},$$

$$L(M) = \bigcup_{x \in M} L_x(M).$$

Define the projection

$$\pi : L(M) \to M \text{ by } u \mapsto x, \text{ where } u \in L_x(M) \subset L(M).$$

The general linear group $GL(m,\mathbf{R})$ *acts freely on* $L(M)$ *on the right by*

$$(u,g) \mapsto ug = (u_i g_1^i, u_i g_2^i, \cdots, u_i g_m^i),\ g = (g_j^i) \in GL(m,\mathbf{R}).$$

It can be shown that $L(M)$ *can be given the structure of a manifold such that this* $GL(m,\mathbf{R})$ *action is smooth. Then* $L(M)(M, GL(m,\mathbf{R}))$ *is a principal bundle over* M *with structure group* $GL(m,\mathbf{R})$. *This principal bundle* $L(M)$ *is called the* bundle of frames *of* M .

Let $P(M,G)$ and $Q(N,H)$ be two principal fiber bundles. A *principal bundle homomorphism* of $Q(N,H)$ into $P(M,G)$ is a bundle homomorphism $f : Q \to P$ together with a Lie group homomorphism $\gamma : H \to G$ such that

$$f(uh) = f(u)\gamma(h),\ \forall u \in Q, h \in H.$$

If f is an imbedding (i.e. a diffeomorphism onto a submanifold) and γ is injective then we say that Q is imbedded in P. Note that in this case the induced morphism

$f_0 : N \to M$ is also an imbedding. If $M = N$ and $f_0 = id_M$, then Q is called a *reduced subbundle* of P or a *reduction of the structure group* G to H where H is regarded as a subgroup of G.

If H is a maximal compact subgroup of G then it can be shown that (D. Husemoller [bHU1], N.E. Steenrod [bST1]) the bundle $P(M,G)$ can be reduced to a bundle $Q(M,H)$. An application of this result to the bundle of frames $L(M)$ (Example 2.2) shows that $L(M)(M, GL(m,\mathbb{R}))$ can be reduced to a subbundle $O(M)$ with structure group $O(m,\mathbb{R})$, the orthogonal group. The bundle $O(M)(M,O(m,\mathbb{R}))$ is called the bundle of orthonormal frames on M. Furthermore, this reduction is equivalent to the existence of a Riemannian structure on M. The structure group $O(m,\mathbb{R})$ of $O(M)$ can be reduced to the special orthogonal group $SO(m,\mathbb{R})$ if and only if the manifold M is orientable. The reduced subbundle of special orthonormal frames is denoted by $SO(M)(M,SO(m,\mathbb{R}))$.

On the other hand in some situations one is interested in extending or lifting the structure group of a bundle to obtain a new principal bundle in the following sense. Let $P(M,G)$ be a principal bundle with structure group G. Let H be a Lie group and let $f : H \to G$ be a surjective, covering homomorphism (see Chapter 4, for an introduction to the topological concepts used in this paragraph) such that $K = Ker\, f \subset Z(H)$, the center of H. We say that $P(M,G)$ has a lift to a principal bundle $Q(M,H)$ with structure group H, if there exists a bundle map

$$\hat{f} : Q \to P \quad \text{such that} \quad \hat{f}(uh) = \hat{f}(u)f(h) \quad \forall u \in Q, h \in H.$$

Using Čech cohomology Greub and Petry [GR2] have shown that there exists a topological obstruction $\eta(P) \in H^2(M,K)$ to the lifting of $P(M,G)$ to $Q(M,H)$, with the property that

$$h \in \mathcal{F}(N,M) \quad \text{implies that} \quad \eta(h^*(P)) = h^*(\eta(P)).$$

An important special case that appears in many applications (see, for example, [SE1], [TA4], [TH2]) is when f is the universal covering map. In this case K is isomorphic to $\pi_1(G)$, the fundamental group of G and hence $\eta(P) \in H^2(M,\pi_1(G))$. The following theorem gives the obstruction $\eta(P)$ in terms of well known characteristic classes for three frequently used groups.

THEOREM 2.1. *In the following H denotes the universal covering group of G.*

1) $G = SO(n)$, $H = Spin(n)$. In this case $\eta(P) = w_2(P) \in H^2(M,\mathbf{Z}_2)$, where $w_2(P)$ is the second Stiefel-Whitney class of P.

2) $G = SO(3,1)_+$, the connected component of the identity of the proper Lorentz group, $H = SL(2,\mathbb{C})$. In this case $\eta(P) = w_2(P) \in H^2(M,\mathbf{Z}_2)$, where $w_2(P)$ is the second Stiefel-Whitney class of P.

3) $G = U(n)$, $H = \mathbf{R} \times SU(n)$ *. In this case* $\eta(P) = c_1(P) \in H^2(M, \mathbf{Z})$ *, where* $c_1(P)$ *is the first Chern class of* P *.* ■

The following example is a typical application of the above theorem.

EXAMPLE 2.3. *Let* M *be an oriented Riemannian manifold. Then its bundle of special orthonormal frames* $SO(M)$ *is a principal* $SO(m, \mathbf{R})$ *bundle. We say that* M *is a spin manifold or that* M *admits a spin structure if the bundle* $SO(M)$ *can be extended to the group* $Spin(m, \mathbf{R})$ *. The principal bundle obtained by this extension is called the* spin (frame-) bundle *and is denoted by* $SP(M)(M, Spin(m, \mathbf{R}))$ *. Let* $SP(M) \times_\rho V$ *be the bundle associated to* $SP(M)$ *by the representation* ρ *of* $Spin(m, \mathbf{R})$ *on the complex vector space* V *. Then* $S_\rho = \Gamma(SP(M) \times_\rho V)$ *is called the space of* spinors of type ρ *. By part 1) of the above theorem we can conclude that* M *admits a spin structure if and only if* $w_2(M) := w_2(SO(M))$ *the second Stiefel-Whitney class of* M *is zero. Topological classification of spin structures is given in* [MI2]*. The mathematical foundations of the theory of spinors were laid by E. Cartan in* [bCA1]*, where the Dirac operator was introduced to study Dirac's equation for the electron. This operator and its various extensions play a fundamental role in the study of the topology and the geometry of manifolds arising in gauge theory (*[AT10]*).*

Let $P(M, G)$ be a principal bundle. The action ρ of G on P induces an injective homomorphism of the Lie algebra $\mathbf{g}$ of G into $\mathcal{X}(P)$ (the Lie algebra of the vector fields on P) as follows. Let $A \in \mathbf{g}$ and let $a_t = exp(tA)$ be the one-parameter subgroup of G generated by A. Restricting the action ρ of G to $exp(tA)$ we get a smooth curve

$$u \cdot exp(tA) := \rho(u, exp(tA))$$

through $u \in P$. The tangent vector to this curve at the point $u \in P$ is denoted by $\tilde{A}_u$. The *fundamental vector field* $\tilde{A} \in \mathcal{X}(P)$ corresponding to $A \in \mathbf{g}$ is defined by the map

$$u \mapsto \tilde{A}_u.$$

One can verify that the map $\tilde{\rho}$ from $\mathbf{g}$ to $\mathcal{X}(P)$ defined by

$$A \mapsto \tilde{A}$$

is an injective Lie algebra homomorphism. We note that, since G acts freely on P, $A \neq 0$ implies $\tilde{A}_u \neq 0$, $\forall u \in P$. Moreover $\tilde{A}$ has another important property, for which we need to give the notion of vertical vector field. $Y \in \mathcal{X}(P)$ is said to be a *vertical* vector field if Y_u is tangent to the fiber of P through u, $\forall u \in P$. We denote by $\mathcal{V}(P)$ the set of vertical vector fields on P. $\mathcal{V}(P)$ is a Lie subalgebra of $\mathcal{X}(P)$. Since G preserves the fibers (i.e. acts vertically on P), $\tilde{A}$ is a vertical vector field and the map

$$\tilde{\rho} : \mathbf{g} \to \mathcal{V}(P) \text{ defined by } A \mapsto \tilde{A}$$

is a Lie algebra isomorphism into $\mathcal{V}(P)$ and induces a trivialization of the vertical bundle $\mathcal{V}(P) \cong P \times \mathbf{g}$.

2.2. Associated bundles

Let $P(M, G)$ be a principal fiber bundle and F be a left G-manifold, i.e. G acts by diffeomorphisms from the left on F. We denote by r this action of G on F and by gf or $r(g)f$ the element $r(g, f)$. Then we have the following right action R of G on the product manifold $P \times F$

$$(u, f)g = (ug, r(g^{-1})f), \ \forall g \in G, (u, f) \in P \times F.$$

The action R is free and the orbit space $(P \times F)/R$ is denoted by $P \times_r F$ or by $E(M, F, r, P)$ (or simply E). We denote by

$$\mathcal{O} : P \times F \to E$$

the quotient map, called the projection onto the orbits of R, and by $[u, f]$ the orbit $\mathcal{O}(u, f)$. We have the following commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{p_1} & P \\ \downarrow{\scriptstyle \mathcal{O}} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi_E} & M \end{array}$$

where $\pi_E : E \to M$ is defined by $[u, f] \mapsto \pi(u)$ and p_1 is the projection onto the first factor. (E, M, π_E, F) is a fiber bundle over M with fiber type F. We call $P \times_r F$ or $E(M, F, r, P)$ the *fiber bundle associated to P with fiber type F.* If F is a vector space then E is called the *vector bundle with fiber type* F *associated to* P.

EXAMPLE 2.4. *Let* Ad *denote the adjoint action of* G *on itself. Then* $P \times_{Ad} G$ *is a bundle of Lie groups associated to* P, *denoted by* $Ad(P)$. *Let* ad *denote the adjoint action of* G *on its Lie algebra* $\mathbf{g}$. *Then* $P \times_{ad} \mathbf{g}$ *is a bundle of Lie algebras associated* *to P, denoted by $ad(P)$. The bundles $Ad(P)$ and $ad(P)$ play a fundamental role in applications to gauge theory.*

Each $u \in P$ induces an isomorphism

$$\tilde{u} : F \to E_{\pi(u)}, \quad \text{defined by } \tilde{u}(f) = \mathcal{O}(u, f),$$

where $\mathcal{O}$ is the orbit projection. We note that the map $\tilde{u}$ satisfies the following relation

$$\widetilde{(ug)}(f) = \tilde{u}(gf), \; \forall g \in G.$$

We denote by $\mathcal{F}_G(P, F)$ the space of G-equivariant maps of P to F, i.e.

$$\mathcal{F}_G(P, F) := \{f : P \to F \mid f(ug) = g^{-1}(f(u))\}.$$

Recall that $\Gamma(E) = \Gamma_M(E(M, F, r, P))$ is the space of smooth sections of E over M. There exists a one-to-one correspondence between $\mathcal{F}_G(P, F)$ and $\Gamma(E)$ which is defined as follows. If $f \in \mathcal{F}_G(P, F)$ we define $s_f \in \Gamma(E)$ by

$$s_f(x) = [u, f(u)], \quad \text{where } u \in \pi^{-1}(x).$$

It is easy to verify that s_f is well defined and that $f \mapsto s_f$ is a one-to-one correspondence from $\mathcal{F}_G(P, F)$ to $\Gamma(E)$ with the inverse defined as follows. If $s \in \Gamma(E)$, then $f_s \in \mathcal{F}_G(P, F)$ is defined by

$$f_s(u) = \tilde{u}^{-1}(s(\pi(u))).$$

EXAMPLE 2.5. *The tangent bundle TM is an associated bundle of $L(M)$ with fiber type $\mathbf{R}^m$ and left action r given by the defining representation of $GL(m, \mathbf{R})$ on $\mathbf{R}^m$, i.e.*

$$TM = E(M, \mathbf{R}^m, r, L(M)).$$

For $u \in L_x(M)$ we have the map

$$\tilde{u} : \mathbf{R}^m \to T_x(M),$$

which is a linear isomorphism. Thus we could define a frame by the map $\tilde{u}$ using the vector bundle structure of TM with the action of $GL(m, \mathbf{R})$ given by the composition

$$\tilde{u} \cdot g = \tilde{u} \circ g, \; \forall g \in GL(m, \mathbf{R})$$

where g is regarded as a map from $\mathbf{R}^m$ to $\mathbf{R}^m$. Similarly it can be shown that the various tensor bundles $T^r_s(M)$ and form bundles $\Lambda^k(M)$ are associated bundles of $L(M)$.

Associated bundles can be used to give the following formulation of the concept of the reduction of the structure group of a principal bundle.

THEOREM 2.2. *If H is a closed subgroup of G, then the structure group G of $P(M,G)$ is reducible to H if and only if the associated bundle $E(M, G/H, q, P(M,G))$ admits a section (where q is the canonical action of G on the quotient G/H).* ■

The theory of fiber bundles occupies a central place in modern mathematics. Standard references are D. Husemoller [bHU1], N.E. Steenrod [bST1].

3. THEORY OF CONNECTIONS

3.1. Connections in a principal fiber bundle and curvature

Let $P(M,G)$ be a principal bundle with structure group G and canonical projection π over a manifold M of dimension m.

DEFINITION 3.1. *A* connection Γ *in $P(M,G)$ is an m-dimensional distibution H on P such that, the following conditions are satisfied for all $u \in P$,*

(i) $T_uP = V_u + H_u$, *where $V_u = Ker(\pi_{*u})$ is the* vertical subspace *of the tangent space T_uP,*

(ii) $H_{\rho_a(u)} = (\rho_a)_* H_u, \forall a \in G$, *where (ρ_a) is the right action of G on P determined by a.*

The condition (ii) *may be rephrased as*

(ii)' *the distribution H is invariant under the action of G on P.*

We call H_u the Γ*-horizontal* subspace of T_uP. Condition (i) allows us to decompose each $X \in T_uP$ into its *vertical part* $v(X) \in V_u$ and the *horizontal part* $h(X) \in H_u$. If $Y \in \mathcal{X}(P)$ is a vector field on P then

$$v(Y) : P \to TP \text{ defined by } u \mapsto v(Y_u)$$

and

$$h(Y) : P \to TP \text{ defined by } u \mapsto h(Y_u)$$

are also in $\mathcal{X}(P)$. We observe that

$$\pi_{*u} : H_u \to T_{\pi(u)}M$$

is an isomorphism. The vector field $Y \in \mathcal{X}(P)$ is said to be Γ*-horizontal* if $Y_u \in H_u$, $\forall u \in P$. The set of horizontal vector fields is a vector subspace but not a Lie subalgebra of $\mathcal{X}(P)$. For $X \in \mathcal{X}(M)$ the Γ*-horizontal lift* (or simply the *lift*) of

X to P is the unique horizontal field $X^h \in \mathcal{X}(P)$ such that $\pi_* X^h = X$. Note that $X^h \in \mathcal{X}(P)$ is invariant under the action of G on P and every horizontal vector field $Y \in \mathcal{X}(P)$ invariant under the action of G is the lift of some $X \in \mathcal{X}(M)$, i.e. $Y = X^h$ for some $X \in \mathcal{X}(M)$. A smooth curve c in P (i.e. $c : I \to P$ is a smooth function from some open interval $I \subset \mathbf{R}$) is called *horizontal* if $\dot{c}(t) \in H_{c(t)}$, $\forall t \in I$. A section $s \in \Gamma(P)$ is called *parallel* if

$$s_*(T_x M) \subset H_{s(x)}, \qquad \forall x \in M.$$

Given a curve $x : [0, 1] \to M$ and $w_0 \in P$ such that $\pi(w_0) = x(0)$, there is a unique *horizontal lift* $w : [0, 1] \to P$ of the curve x such that

$$\pi(w(t)) = x(t), \forall t \in [0, 1],$$

i.e. the curve w in P is horizontal and $\pi(w(t)) = x(t)$. Given a closed curve x at $p \in M$, i.e. $p = x_0 = x_1$, the horizontal lift induces an isomorphism of the fiber $\pi^{-1}(p)$. The set of all such isomorphisms forms a group called the *holonomy group* of Γ at $p \in M$. It can be shown that the holonomy group is isomorphic to a Lie subgroup of the structure group G.

If $X, Y \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$, then the following properties are easily verified:

(i) $$X^h + Y^h = (X + Y)^h,$$

(ii) $$(\pi^* f) X^h = (f X)^h,$$

(iii) $$h([X^h, Y^h]) = [X, Y]^h.$$

We recall that a k-form α on P with values in the vector space V is a map

$$u \mapsto \alpha_u,$$

where

$$\alpha_u : \underbrace{T_u P \times \cdots \times T_u P}_{k \ times} \to V$$

is a multilinear anti-symmetric map. Given a basis $\{v_i\}_{1 \le i \le n}$ in V, we can express α as the formal sum

$$\alpha = \sum_{i=1}^{n} \alpha_i v_i,$$

where $\alpha_i \in \Lambda^k(P)$, $\forall i$. We define a 1-form $\omega \in \Lambda^1(P, \mathbf{g})$ on P with values in the Lie algebra $\mathbf{g}$ by using the connection Γ as follows

$$\omega(X) = \tilde{\rho}^{-1}(v(X)),$$

where $\tilde{\rho} : \mathbf{g} \to \mathcal{V}(P)$ is the isomorphism defined at the end of section 2.1. The 1-form ω is called the *connection 1-form* of the connection Γ. It can be shown that the connection 1-form ω satisfies the following conditions (3.1) and (3.2)

(3.1) $$\omega(\tilde{A}) = A, \ \forall A \in \mathbf{g},$$

(3.2) $$(\rho_a)^* \omega = ad(a^{-1})\omega, \ \forall a \in G.$$

Condition (3.2) means that ω is G-equivariant. Conditions (3.1) and (3.2) characterize the connection ω on P. In fact one may give the following alternative definition of a connection.

DEFINITION 3.2. *A connection in $P(M,G)$ is a 1-form $\omega \in \Lambda^1(P, \mathbf{g})$ which satisfies the conditions (3.1) and (3.2) given above.*

Note that given a 1-form ω satisfying the conditions (3.1) and (3.2) we can define the distribution $H : u \to H_u$ on P by

(3.3) $$H_u := \{Y \in T_u P \mid \omega_u(Y) = 0\}.$$

One can then verify that the distribution H is m-dimensional and defines a connection Γ according to the definition 3.1 and the connection 1- form associated to Γ is ω.

We now motivate a third definition of a connection in terms of a local representation of the bundle $P(M,G)$. Let ω be a connection on $P(M,G)$. Let $\{(U_i, \psi_i)\}_{i \in I}$ be a local representation of $P(M,G)$ with transition functions

$$\psi_{ij} : U_{ij} \to G.$$

Let $e \in G$ be the identity element of G and let $s_i : U_i \to P$ be a local section defined by

$$s_i(x) = \psi_i(x, e).$$

Define the family $\{\omega_i\}_{i \in I}$, of 1-forms

$$\omega_i \in \Lambda^1(U_i, \mathbf{g}) \quad \text{by} \quad \omega_i = s_i^*(\omega)$$

where ω is a connection on P. Let $\Theta \in \Lambda^1(G, \mathbf{g})$ be the canonical 1-form of G; then writing $\Theta_{ij} = \psi_{ij}^* \Theta$, we obtain the following relations

(3.4) $$\omega_j(x) = ad(\psi_{ij}(x))^{-1} \omega_i(x) + \Theta_{ij}(x), \ \forall x \in U_{ij} \text{ and } \forall i, j \in I.$$

Thus, given a connection ω on P, we have a family $\{(U_i, \psi_i, \omega_i)\}_{i\in I}$ where $\{(U_i, \psi_i)\}_{i\in I}$ is a local representation of P and $\{\omega_i\}_{i\in I}$ is a family of 1-forms satisfying the conditions (3.4). Vice versa, given such a family $\{(U_i, \psi_i, \omega_i)\}_{i\in I}$ satisfying the conditions (3.4), this determines a connection in the following way. Let $\psi_i^{-1} : \pi^{-1}(U_i) \to U_i \times G$ and let p_1, p_2 be the canonical projections of $U_i \times G$ on the first and second factors respectively. Let

$$\alpha_i = (p_1 \circ \psi_i^{-1})^*\omega_i + (p_2 \circ \psi_i^{-1})^*\Theta.$$

Then $\alpha_i \in \Lambda^1(\pi^{-1}(U_i), \mathrm{g})$. Define $\omega \in \Lambda^1(P, \mathrm{g})$ by $\omega = \alpha_i$ on $\pi^{-1}(U_i)$. This is well defined because on the intersection $\pi^{-1}(U_{ij})$ the conditions (3.4) guarantee that $\alpha_i = \alpha_j$. Then one can show that ω is a connection according to the definition 3.2. Thus we may give the following third definition of a connection equivalent to the two definitions given above.

DEFINITION 3.3. *A* connection *in $P(M,G)$ is a family of triples $\{(U_i, \psi_i, \omega_i)\}_{i\in I}$, where $\{(U_i, \psi_i)\}_{i\in I}$ is a local representation of P and $\{\omega_i\}_{i\in I}$ is a family of 1-forms satisfying the relations* (3.4).

Frequently it is this local definition that is used to construct a connection.

EXAMPLE 3.1. *Let M be a paracompact manifold and let $P = L(M)$ be the bundle of frames on M. Recall that M admits a Riemannian metric so that the bundle of frames $L(M)$ can be reduced to a bundle of orthonormal frames. By using the local isomorphism with R^m and pulling back the flat connection to $L(M)$, we obtain the so called Riemannian connection in $L(M)$.*

Let $\phi \in \Lambda^k(P, V)$ be a k-form in P with values in a vector space V. Let $r : G \to GL(V)$ be a representation of G on V. We say that ϕ is *pseudo-tensorial* of type (r, V) if

$$\rho_a^*\phi = r(a^{-1}) \cdot \phi, \ \forall a \in G.$$

A connection 1-form ω on P is pseudo-tensorial of type (ad, g). The form $\phi \in \Lambda^k(P, V)$ is called *horizontal* if

$$\phi(X_1, X_2, \cdots, X_k) = 0$$

if some X_i, $1 \le i \le k$ is vertical. The form $\phi \in \Lambda^k(P, V)$ is called *tensorial* of type (r, V) if it is horizontal and pseudo-tensorial. If the k-form $\phi \in \Lambda^k(P, V)$ is tensorial of type (r, V) then there exists a unique k-form s_ϕ on M with values in the vector bundle $E = P \times_r V$ defined as follows:

$$s_\phi(x)(X_1, X_2, \cdots, X_k) = \tilde{u}\phi(u)(Y_1, Y_2, \cdots, Y_k), \ \forall x \in M$$

where $u \in \pi^{-1}(x)$ and $Y_i \in T_u P$ such that $T\pi(Y_i) = X_i$, $1 \leq i \leq k$. Due to tensoriality of ϕ this definition of s_ϕ is independent of the choice of u and Y_i. We call $s_\phi \in \Lambda^k(M, E)$ the k-form *associated to* ϕ. We note that the one-to-one correspondence $f \mapsto s_f$ between $\mathcal{F}_G(P, F)$ and $\Gamma(E)$ defined in section 2.2 extends to a one-to-one correspondence $\phi \mapsto s_\phi$ of tensorial forms of type (r, V) and forms with values in the vector bundle E defined above.

Given a connection 1-form ω on P we define

$$d^\omega : \Lambda^k(P, \mathbf{g}) \to \Lambda^{k+1}(P, \mathbf{g})$$

by

$$d^\omega \alpha(X_0, \cdots, X_k) = d\alpha(h(X_0), \cdots, h(X_k)), \ \forall \alpha \in \Lambda^k(P, \mathbf{g}).$$

We define the *curvature* 2-form $\Omega \in \Lambda^2(P, \mathbf{g})$ by

(3.5) $$\Omega := d^\omega \omega.$$

It is easy to verify that Ω is a tensorial 2-form of type $(ad, \mathbf{g})$ and that it satisfies the following conditions.

(3.6) $$d\omega(X, Y) = \Omega(X, Y) - \frac{1}{2}[\omega(X), \omega(Y)].$$

Equation (3.6) is called the *structure equation* and is often written in the form

$$d\omega = \Omega - \omega \wedge \omega.$$

Using the definition (3.5) we obtain the following *Bianchi identities*

(3.7) $$d^\omega \Omega = 0.$$

There exists a unique 2-form $F_\omega \in \Lambda^2(M, ad(P))$ associated to Ω so that

(3.8) $$F_\omega = s_\Omega.$$

The 2-form F_ω is called the *curvature 2-form* on M corresponding to the connection ω on P.

Linear connections

Let $L(M)$ be the bundle of frames of M. Then we have seen that $L(M)$ is a principal bundle with structure group $GL(m, \mathbf{R})$. A connection on this principal bundle is called a *linear connection*. If M is a Riemannian manifold, then the structure group $GL(m, \mathbf{R})$ can be reduced to the orthogonal group $O(m, \mathbf{R})$. A connection on the reduced bundle is called a *Riemannian connection*. Similarly if M is a Lorentz manifold, then the structure group $GL(m, \mathbf{R})$ can be reduced to the Lorentz group. The frames in the reduced bundle are called *local inertial frames* and the connection on the reduced bundle is called a *Lorentz connection*. If $m = 4$ then the Lorentz manifold M is called a *space-time manifold*. The connection and curvature can be interpreted as representing gravitational potential and field.

Recall that the tangent bundle $TM = E(M, \mathbf{R}^m, GL(m, \mathbf{R}), L(M))$ is a vector bundle associated to the bundle of frames $L(M)$. In particular, a frame $u \in L(M)$ induces an isomorphism

$$\tilde{u} : \mathbf{R}^m \to T_{\pi(u)} M.$$

Let $X \in T_u L(M)$ and define the 1-form $\theta \in \Lambda^1(L(M), \mathbf{R}^m)$ by

$$\theta_u(X) = \tilde{u}^{-1}(\pi_*(X)).$$

Then θ is a tensorial 1-form on $L(M)$ of type $(\tau, \mathbf{R}^m)$ where τ is the defining representation of $GL(m, \mathbf{R})$. In the physics literature the form θ is frequently called the *soldering form*. If we choose a basis for the Lie algebra $gl(m, \mathbf{R})$ and a basis for $\mathbf{R}^m$, then we can express the connection 1-form ω as m^2 1-forms ω_{ij} and the form θ as m 1-forms θ_k, $1 \leq k \leq m$. These $m^2 + m$ forms are globally defined on $L(M)$ and make $L(M)$ into a parallelizable manifold. In particular we can define a Riemannian metric on $L(M)$ by

$$ds^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \omega_{ij}^2 + \sum_{k=1}^{m} \theta_k^2.$$

Using this Riemannian metric we can make $L(M)$ into a topological metric space. This metric can be used to show that a manifold M which admits a linear connection (in particular, a Lorentz connection) must be a topological metric space. Thus in considering manifolds of interest in physical applications one may restrict to the manifolds which are metric spaces. The metric on $L(M)$ is called the bundle metric or b-metric and is used in defining the so called b-boundaries for space-time manifolds (for details see [MA5], [SC1]).

We observe that the linear connections are distinguished from connections in other principal bundles by the existence of the soldering form. Thus in addition to the curvature 2-form $\Omega = d^{\omega}\omega$, we have the *torsion* 2-form $\Theta = d^{\omega}\theta \in \Lambda^2(L(M), \mathbf{R}^m)$. A connection ω is called *torsion-free* if the torsion 2-form $\Theta = d^{\omega}\theta = 0$. Among all linear connections on a pseudo-Riemannian manifold there exists a unique torsion-free connection λ called the *Levi-Civita connection*. The Levi-Civita connection is also referred to as the symmetric connection. The curvature 2-form F_λ of the Levi-Civita connection λ is denoted by R and is called the *Riemann curvature* of M. Let us suppose that M is an oriented Riemannian manifold. The identification of the Lie algebra $so(m)$ with $\Lambda^2(\mathbf{R}^m)$ allows us to identify $ad\, L(M)$ with $\Lambda^2(M)$. Thus for each $x \in M$, R defines a symmetric, linear transformation of $\Lambda^2_x(M)$. The dimension 4 is further distinguished by the fact that $so(4) = so(3) \oplus so(3)$ and that this decomposition corresponds to the decomposition $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$ into ± 1 eigen-spaces of the Hodge star operator. The Riemann curvature R also decomposes into $SO(4)$-invariant components induced by the above direct sum decomposition of $\Lambda^2(M)$. These components are the self-dual (anti-self-dual) Weyl tensor W^+ (W^-), the trace-free part of the Ricci tensor K, and the scalar curvature S. These components can be used to define several important classes of 4-manifolds. Thus M is called a self-dual (anti-self-dual) manifold if $W^- = 0$ ($W^+ = 0$). It is called conformally flat if it is both self-dual and anti-self-dual or if the full Weyl tensor W is zero. M is called an Einstein manifold if the trace-free part of the Ricci tensor $K = 0$. Einstein manifolds correspond to a class of gravitational instantons [MA8]. Einstein manifolds were characterized in [SI7] by the commutation condition $[R, *] = 0$, where the Riemann curvature R and the Hodge star operator $*$ are both regarded as linear tranformations of $\Lambda^2(M)$. This condition was generalized in [MA6], [MA7] to obtain a new formulation of the gravitational field equations. For further details on Riemannian geometry in dimension 4 see [bBE2]. For the study of Riemannian manifolds and manifolds of differentiable mappings see J. Cheeger, D. Ebin [bCH1], P.W. Michor [bMI1].

3.2. Connections on associated bundles and covariant derivatives

Let $P(M,G)$ be a principal bundle and $E(M,F,r,P)$ be the associated fiber bundle over M with fiber type F and action r. A connection Γ in P allows us to define the notion of a horizontal vector field on E. Let $w \in E$ and $(u,a) \in \mathcal{O}^{-1}(w)$, where $\mathcal{O} : P \times F \to E$ is the canonical orbit projection. Define

$$f_a : P \to E \text{ by } u \mapsto \mathcal{O}(u,a),$$

where $\mathcal{O}(u,a)$ is the orbit of (u,a) in E. Now define $H_w = H_w E \subset T_w E$ by

$$H_w := (f_a)_*(H_u P),$$

where H_uP is the horizontal subspace of T_uP. It can be shown that H_wE is independent of the choice of $(u,a) \in \mathcal{O}^{-1}(w)$ and is thus well defined. A vector field $X \in \mathcal{X}(E)$ is called *horizontal* if $X_w \in H_w$, $\forall w \in E$. A smooth curve c in E (i.e. $c : I \to E$ is a smooth function from some open interval $I \subset \mathbf{R}$) is called *horizontal* if $\dot{c}(t) \in H_{c(t)}$, $\forall t \in I$. A section $s \in \Gamma(E)$ is called *parallel* if

$$s_*(T_xM) \subset H_{s(x)}, \qquad \forall x \in M.$$

Given a curve $x : [0, 1] \to M$ and $w_0 \in E$ such that $\pi_E(w_0) = x(0)$, there is a unique *horizontal lift* $w : [0, 1] \to E$ of the curve x such that

$$\pi_E(w(t)) = x(t), \forall t \in [0, 1],$$

i.e. the curve w in E is horizontal and $\pi_E(w(t)) = x(t)$. Given the curve x, joining x_0 and x_1, the horizontal lift induces a diffeomorphism of $\pi_E^{-1}(x_0)$ and $\pi_E^{-1}(x_1)$ called *parallel translation* or *parallel displacement* of fibers of E. In particular if E is a vector bundle (i.e. the fiber type F is a vector space), then parallel displacement is an isomorphism and we may define the *covariant derivative* $\nabla_{\dot{x}(t)}s$ of a section s at $x(t)$ along the vector $\dot{x}(t)$ by the formula

$$\nabla_{\dot{x}(t)}s = \lim_{h \to 0} \frac{1}{h}[c_{t,t+h}^{-1}(s(x(t+h))) - s(x(t))],$$

where

$$c_{t,t+h} : \pi_E^{-1}(x(t)) \to \pi_E^{-1}(x(t+h))$$

is parallel displacement along x from $x(t)$ to $x(t+h)$. If $X \in \mathcal{X}(M)$ and x is the integral curve of X through x_0, so that $X_{x(t)} = \dot{x}(t)$, then the above definition may be used to define the covariant derivative $\nabla_X s$. The operator $\nabla_X : \Gamma(E) \to \Gamma(E)$ satisfies the following relations, $\forall X, Y \in \mathcal{X}(M)$, $\forall f \in \mathcal{F}(M)$, and $\forall, t, s \in \Gamma(E)$

(a) $\nabla_{X+Y}s = \nabla_X s + \nabla_Y s,$

(b) $\nabla_X(s+t) = \nabla_X s + \nabla_X t,$

(c) $\nabla_{fX}s = f\nabla_X s,$

(d) $\nabla_X(fs) = f\nabla_X s + (Xf)s.$

We note that the covariant derivative may be defined in terms of the Lie derivative as follows. We recall first (see section 2.2) that there is a one-to-one correspondence between G-equivariant functions from P to F and sections in $\Gamma(E)$, which is defined as follows. Let $s \in \Gamma(E)$; then we define

$$f_s : P \to F \text{ by } u \mapsto \tilde{u}^{-1}(s(x)),$$

where $\pi(u) = x$ and $\tilde{u} : F \to E_x$ is the isomorphism defined by $u \in P$. Conversely given a G-equivariant map f from P to F define $s_f \in \Gamma(E)$ by

$$s_f(x) = \mathcal{O}(u, f(u)),$$

where $u \in \pi^{-1}(x)$. This is well defined because of equivariance. The covariant derivative $\nabla_X s$ corresponds to $L_{\hat{X}}(f_s)$ where f_s is defined above and $\hat{X}$ is the horizontal lift of X to P.

The covariant derivative defines a map ∇ from $\Gamma(E)$ to $\Gamma(T^*(M) \otimes E)$ as follows. $\nabla s : M \to T^*(M) \otimes E$ is a map such that

$$\nabla s(x) \cdot X_x = (\nabla_X s)(x).$$

We have

$$\nabla(fs) = df \otimes s + f\nabla s, \quad \forall f \in \mathcal{F}(M).$$

Recall that a section $s \in \Gamma(T^*(M) \otimes E)$ may be regarded as defining, for each $x \in M$, a map $s(x) : T_x M \to E_x$. To indicate the dependence of ∇ on the connection 1-form ω, it is customary to denote it by d^ω, in agreement with the notation used for vector valued forms. We extend the operator d^ω to $\Lambda^p(M, E)$ as follows

$$d^\omega \alpha(X_0, \cdots, X_p) =$$
$$= \sum_{j=0}^{p} (-1)^j \nabla_{X_j}(\alpha(X_0, \cdots, \hat{X}_j, \cdots, X_p)) +$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p),$$

where the hat sign «^» on a vector field denotes deletion of that vector field. The operator d^ω satisfies the following relation.

$$d^\omega(\beta \wedge \alpha) = (d^\omega \beta) \wedge \alpha + (-1)^{deg(\beta)} \beta \wedge (d^\omega \alpha),$$
$$\alpha \in \Lambda^*(M, E), \beta \in \Lambda^*(M, \mathbf{R}).$$

In fact, this relation can be used to define the operator d^ω. It is customary to write d^ω_p for the above operator if we want to emphasize its action on p-forms, otherwise we write d^ω t o denote any one of these operators. Thus our earlier definition of d^ω : $\Lambda^0(M, E) \to \Lambda^1(M, E)$, combined with the above definition, gives us the following sequence

$$0 \xrightarrow{id} \Lambda^0(M, E) \xrightarrow{d^\omega} \Lambda^1(M, E) \xrightarrow{d^\omega} \Lambda^2(M, E) \xrightarrow{d^\omega} \cdots$$

which is called the *generalized deRham sequence*. We will comment on this in section 4.2. We will find these concepts useful in the study of gauge fields and their associated fields.

3.3. Generalized connections

The various definitions of connection in principal and associated bundles given in sections 3.1, 3.2 above are adequate for most of the applications discussed in this paper. However, it is possible to define the notion of connection on an arbitrary fiber bundle. We call this a generalized connection and use it to give an alternative formulation of some aspects of gauge theories. There is extensive work in this area. We cite only a few references as examples: [GA2], [GA3], [MA1], [MA2], [MA3], [MO1], [PE1]. Let E be a fiber bundle over B with projection $p : E \to B$. Let VE denote the vertical vector bundle over E. VE is a subbundle of the tangent bundle TE, the fiber V_uE, $u \in E$ being the tangent space to the fiber $E_{p(u)}$ of E passing through u. Let $p^*(TB)$ be the pull back of the tangent bundle of B to E. Then we have the following short exact sequence of vector bundles and morphisms over E:

$$0 \to VE \xrightarrow{i} TE \xrightarrow{a} \pi^*(TB) \to 0 \tag{3.9}$$

where i is the injection of the vertical bundle into the tangent bundle, and a is defined by $(e, X_e) \mapsto (e, p_*(X_e))$. We define a generalized connection on E to be a splitting of the above exact sequence, i.e. a vector bundle morphism $c : \pi^*(TB) \to TE$ such that $a \circ c = id_{\pi^*(TB)}$ Let $b \in B$ and let $e \in p^{-1}(b)$. Then the splitting c induces an injection $\hat{c}_e : T_b(B) \to T_e(E)$, $X \mapsto c(e, X)$. We call $\hat{c}_e(X)$ the *horizontal lift* of the tangent vector X to $e \in E$. We call $\hat{c}_e(T_b(B))$ the *horizontal space at* $e \in E$ and denote it by H_eE. The spaces H_eE are the fibers of the *horizontal bundle* HE and we have the decomposition

$$TE = HE \oplus VE.$$

We note that this decomposition corresponds to the condition (i) of Definition 3.1. Alternatively, a connection may be defined as a section of the first jet bundle $J^1(E)$ over E. The definitions of covariant derivative, covariant differential and curvature can be formulated in this general context. An introduction to this approach and its physical applications is given in [MA2], [MA3], [MO1].

If $E = P(M, G)$, then we can recover the usual definition of connection as follows. The action of G on P extends to all the vector bundles in the exact sequence (3.9) to give the following short exact sequence of vector bundles over M:

$$0 \to VP/G \xrightarrow{i} TP/G \xrightarrow{\hat{a}} \pi^*(TM)/G \to 0$$

We can rewrite the above sequence as follows:

$$0 \to ad(P) \xrightarrow{i} TP/G \xrightarrow{\hat{a}} TM \to 0 \tag{3.10}$$

A connection on P is then defined as a splitting of the short exact sequence (3.10). This splitting induces a splitting of the sequence (3.9) and we recover Definition 3.1 of the connection given earlier. This approach helps to clarify the role of the structure group in the usual definition of connection in a principal bundle.

4. HOMOTOPY AND COHOMOLOGY OF A MANIFOLD

4.1. Homotopy and classifying spaces

Let X be a Hausdorff topological space. A *path* in X from $x \in X$ to $y \in X$ is a continuous map $\alpha : I = [0,1] \to X$, such that $\alpha(0) = x$, $\alpha(1) = y$. We say that X is *path connected* if there exists a path from x to y for any $x, y \in X$. X is *locally path connected* if its topology is generated by path connected open sets. From now on we assume all topological spaces to be path connected and locally path connected.

Let $a, b \in X$ be two fixed points. Let $P_{a,b}$ be the set of paths in X from a to b. For $\alpha \in P_{a,b}$, we define the *opposite path* $\overleftarrow{\alpha} \in P_{b,a}$, by $\overleftarrow{\alpha}(t) = \alpha(1-t)$, $\forall t \in I$. Let $\alpha, \beta \in P_{a,b}$; we say that α is *homotopic* to β (written as $\alpha \sim \beta$) if there exists a continuous function $H : I \times I \to X$ such that the following conditions hold:

$$H(t,0) = \alpha(t), \quad H(t,1) = \beta(t), \quad \forall t \in I,$$

$$H(0,s) = a, \quad H(1,s) = b, \quad \forall s \in I.$$

H is called a *homotopy* between α and β. We may think of H as a family of paths from a to b parametrized by s, which deforms the path α continuously into the path β. It can be shown that the relation of homotopy between paths is an equivalence relation. We denote the equivalence class of α by $[\alpha]$. The set of all equivalence classes of paths in $P_{a,b}$ is denoted by $E_{a,b}$. If $\alpha \in P_{a,a}$, then we call α a *loop* at a. If $\alpha, \beta \in P_{a,a}$, then we denote by $\alpha * \beta \in P_{a,a}$ the loop defined by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1), & 1/2 \le t \le 1. \end{cases}$$

The operation $*$ induces an operation on $E_{a,a}$ which we denote by juxtaposition. This operation makes $E_{a,a}$ into a group with identity the class $[c_a]$ of the constant loop at a, the class $[\overleftarrow{\alpha}]$ being the inverse of $[\alpha]$. This group is called the *fundamental group* or the *first homotopy group* of X at a and is denoted by $\pi_1(X,a)$. If X is path connected then $\pi_1(X,a) \cong \pi_1(X,b), \forall a, b \in X$. In view of this result we sometimes write $\pi_1(X)$ for the fundamental group of a space X. We say that X is *simply connected* if $\pi_1(X)$ is the trivial group consisting of only the identity element. The fundamental group is an important invariant of a topological space, i.e. $X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$. A surprising application of the non-triviality of the fundamental group is found in the Bohm-Aharanov effect in abelian gauge theories, which we will discuss in section 6.1.

Let $f : (X, x) \to (Y, y)$ be a map of pointed spaces, i.e. $f : X \to Y$ is continuous and $f(x) = y$. Homotopy of maps of pointed spaces is defined similarly. The map f induces a homomorphism

$$\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y),$$

defined by $[\alpha] \mapsto [f \circ \alpha]$. If $f \sim g$ then $\pi_1(f) = \pi_1(g)$. We say that X is *contractible* if $id_X \sim c_x$, the constant map. We note that a contractible space is simply connected. (X, x) is said to be *homotopically equivalent* to or of the same *homotopy type* as (Y, y) if there exist maps $f : (X, x) \to (Y, y)$ and $g : (Y, y) \to (X, x)$ such that

$$f \circ g \sim id_Y, \qquad g \circ f \sim id_X.$$

The relation of homotopic equivalence is, in general, weaker than homeomorphism. One of the outstanding conjectures due to Poincaré is the following.

POINCARÉ CONJECTURE. *Every closed (i.e. compact and without boundary) simply connected 3-manifold is homeomorphic to* S^3.

This conjecture is as yet unsettled. With 3 replaced by $n > 3$ the above conjecture is not true as we will see below. However, we have the following:

GENERALIZED POINCARÉ CONJECTURE. *Every closed n-manifold homotopically equivalent to the n-sphere S^n is homeomorphic to* S^n.

This generalized conjecture was proved to be true for $n > 4$ by Smale in 1960. The case $n = 4$ was settled in the affirmative by Freedman [FR1] in 1980.

Let $p : E \to B$ be a continuous surjection. We say that the pair (E, p) is a *covering* of B if each $x \in B$ has a path-connected neighbourhood U such that each pathwise connected component of $p^{-1}(U)$ is homeomorphic to U. In particular p is a local homeomorphism. E is called the *covering space*, B the *base space* and p the *covering projection.* It can be shown that the cardinality of the fibers $p^{-1}(x), x \in B$ is the same for all x. If this cardinality is a natural number n, then we say that (E, p) is an n-fold covering of B.

EXAMPLE 4.1.

(1) Let $q_n : U(1) \to U(1)$ *be the map defined by*

$$q_n(z) = z^n$$

where $z \in U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. *Thus* $(U(1), q_n)$ *is a n-fold covering of* $U(1)$.

(2) Let $p : \mathbf{R} \to U(1)$ *be the map defined by*

$$p(t) = exp(2\pi i t).$$

In this case the fiber $p^{-1}(1)$ *is* $\mathbf{Z}$.

(3) Let $\pi : S^n \to \mathbf{RP}^n$ *be the natural projection*

$$x \mapsto [x].$$

This is a 2*-fold covering. For* $n = 3$, $S^3 \cong SU(2)$ *and* $\mathbf{RP}^3 \cong SO(3)$. *This is well known in physics as associating two spin matrices in* $SU(2)$ *to the same angular momentum matrix in* $SO(3)$.

A covering space (U, q) with U simply connected is called a *universal covering space* of the base space B. If (E, p) is a covering of B and (U, q) is a universal covering of B then there exists a unique continuous surjection $f : U \to E$ such that (U, f) is a covering of E and the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ & {}_q\searrow \quad \swarrow_p & \\ & B & \end{array}$$

From this it follows that, if $(U_1, q_1), (U_2, q_2)$ are two universal covering spaces of B, then there exists a unique isomorphism f of U_1 onto U_2 such that the following diagram commutes

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_2 \\ & {}_{q_1}\searrow \quad \swarrow_{q_2} & \\ & B & \end{array}$$

Let (U, q) be a universal covering of B. A *covering* or *deck transformation* f is an automorphism of U such that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ & {}_q\searrow \quad \swarrow_q & \\ & B & \end{array}$$

It can be shown that the set $C(U, q)$ of covering transformations is a subgroup of $Aut(U)$ and it is isomorphic to $\pi_1(B)$. If X, Y are topological spaces, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. In particular, if X and Y are simply connected, then $X \times Y$ is simply connected.

EXAMPLE 4.2. *The covering* $(U(1), q_n)$ *of Example 4.1(1) above is not a universal covering while the coverings* $(\mathbf{R}, p)$ *of Example 4.1(2) and* $(SU(2), \pi)$ *of Example 4.1(3) are universal coverings. From this it is easy to deduce the following*

$$\pi_1(\mathbf{RP}^1) \cong \pi_1(S^1) \cong \pi_1(U(1)) \cong \mathbf{Z},$$

$$\pi_1(T^n) \cong \mathbf{Z}^n$$

where

$$T^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$$

is the real n *torus,*

$$\pi_1(S^{n+1}) \cong id, \quad \pi_1(\mathbf{R}^n) \cong id, \quad n \geq 1$$

$$\pi_1(\mathbf{RP}^n) \cong \mathbf{Z}_2, \quad n > 1.$$

If B is a manifold then there exists a universal covering (U, q) of B such that U is also a manifold and q is smooth. If G is a Lie group then there exists a universal covering (U, p) of G such that U is a simply connected Lie group and p is a local isomorphism of Lie groups. In particular G and all its covering spaces are locally isomorphic Lie groups and hence have the same Lie algebra. This fact has the following application in representation theory. Given a representation r of a Lie algebra L on V, there exists a unique simply connected Lie group U with Lie algebra $u \cong L$ and a representation ρ of U on V such that its induced representation $\hat{\rho}$ of u on V is equivalent to r. If G is a Lie group with Lie algebra $\mathbf{g} \cong L$, then we get a representation of $\mathbf{g}$ on V only if the representation ρ of U is equivariant under the action of $\pi_1(G)$. Thus from a representation r of the angular momentum algebra $so(3)$ we get a unique representation of the group $Spin(3) \cong SU(2)$ (spin representation). However, r gives a representation of $SO(3)$ (an angular momentum representation) only for even parity, r in this case, being invariant under the action of $\pi_1(SO(3)) \cong \mathbf{Z}_2$. A similar situation arises for the case of the Lorentz group $SO(3,1)$ and its universal covering group $SL(2, \mathbf{C})$.

There are several possible ways to generalize the definition of π_1 to obtain the higher homotopy groups. We list three important approaches.

(1). Let

$$I^n = \{t = (t_1, \ldots, t_n) \in \mathbf{R}^n \mid 0 \leq t_i \leq 1, \quad 1 \leq i \leq n\}.$$

Define the boundary of I^n by

$$\partial I^n = \{\, t \in \mathbf{R}^n \mid t_i = 0 \text{ or } t_i = 1 \text{ for some } i,\ 1 \leq i \leq n\}.$$

Define

$$F_n(X, x_0) = \{ \alpha : I^n \to X \mid \alpha(\partial I^n) = x_0 \}.$$

We say that $\alpha \sim \beta$ if there exists a homotopy $H : I^n \times I \to X$ such that

$$H(t,0) = \alpha, \quad H(t,1) = \beta$$
$$H(t,s) = x_0, \quad \forall t \in \partial I^n, \; s \in I.$$

For $\alpha, \beta \in F_n(X, x_0)$ we define $\alpha * \beta \in F_n(X, x_0)$ by applying our previous definition for paths to the first coordinate t_1 i.e.

$$\alpha * \beta(t) = \begin{cases} \alpha(2t_1, t_2, \ldots, t_n) & 0 \le t_1 \le 1/2 \\ \beta(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \le t_1 \le 1 \end{cases}$$

Homotopy is an equivalence relation on $F_n(X, x_0)$ and we denote the set of equivalence classes by $E_n(X, x_0)$. It can be shown that $E_n(X, x_0)$ with the operation induced by $*$ is a group. It is called the $n\!-\!th$ *homotopy group* of X at x_0 and is denoted by $\pi_n(X, x_0)$.

(2). The second definition is obtained by identifying the boundary ∂I^n to a fixed point say $e_1 = (1, 0, \ldots, 0) \in S^n$. The quotient space $I^n / \partial I^n \cong S^n$. Hence we may consider $\alpha : (S^n, e_1) \to (X, x_0)$, with the definition of homotopy H between α and β given by $H : S^n \times I \to X$ such that

$$H(x,0) = \alpha, \quad H(x,1) = \beta, \quad \forall x \in S^n$$
$$H(e_1, s) = x_0, \quad \forall s \in I.$$

The group $\pi_n(X, x_0)$ is then defined by an obvious modification of definition (1).

(3). This definition considers the loops on the space of loops. Loop spaces have recently arisen in many physical calculations. We therefore give an indication of the construction of $\pi_2(X, x_0)$. Let $\Omega(X, x_0)$ be the set of all continuous loops in X at x_0. Let $c_{x_0} \in \Omega(X, x_0)$ be the constant loop at x_0. We define $\pi_2(X, x_0) = \pi_1(\Omega(X, x_0), c_{x_0})$.

For a detailed discussion of these three definitions and their applications see, for example, F. Croom [bCR1].

We collect together some properties of the groups π_n.

(i) $\pi_n(X, x_0) = \pi_n(X, x_1)$, $\forall x_0, x_1 \in X$. In view of this we will write $\pi_n(X)$ instead of $\pi_n(X, x_0)$.

(ii) If X is contractible by a homotopy leaving x_0 fixed then $\pi_n(X) = id$.

(iii) $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$.

(iv) $\pi_n(X)$ is abelian for $n > 1$.

(v) If (E,p) is a covering space of X, then p induces an isomorphism $p_* : \pi_n(E) \to \pi_n(X)$ for $n > 1$.

(vi) (Freudenthal) There exists a homomorphism $E : \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$, called the *suspension homomorphism*, with the following properties:

(a) E is surjective for $k = 2n-1$,

(b) E is an isomorphism for $k < 2n-1$.

EXAMPLE 4.3. *(1)* $\pi_n(\mathbf{R}^m) \cong id$.

(2) $\pi_k(S^n) \cong id,\ k < n$.

(3) $\pi_n(S^n) = \mathbf{Z}$.

From (2) and (3) it follows that $\pi_2(S^4) \cong id$ *and*

$$\pi_2(S^2 \times S^2) = \pi_2(S^2) \times \pi_2(S^2) = \mathbf{Z} \times \mathbf{Z}.$$

Also $\pi_1(S^4) \cong id$ *and* $\pi_1(S^2 \times S^2) = \pi_1(S^2) \times \pi_1(S^2) = id$. *Thus* S^4 *and* $S^2 \times S^2$ *are both closed simply connected manifolds which are not homeomorphic. This illustrates the role that higher homotopy groups play in the generalized Poincaré conjecture.*

(4) $\pi_3(G) \cong Z$ *where* G *is any one of the groups* $SO(n), O(n), SU(n), U(n)$. *An element* $\alpha \in \pi_3(G)$ *is often referred to as the* topological quantum number *or the* instanton number, *since it is used to classify* G*-instantons on* S^4. *This example will be discussed in detail in sections 6.2 and 6.3.*

*(5) (*Hopf fibering*) An important example of computation of higher homotopy groups was given by H. Hopf in 1931 in his computation of* $\pi_3(S^2)$. *Consider the following action* $h : U(1) \times \mathbf{C}^2 \to \mathbf{C}^2$ *of* $U(1)$ *on* $\mathbf{C}^2$ *defined by*

$$(z,(z_1,z_2)) \mapsto (zz_1, zz_2).$$

This action leaves the unit sphere $S^3 \subset \mathbf{C}^2$ *invariant and hence induces an action on* S^3 *with fibers isomorphic to* S^1 *and quotient* $\mathbf{CP}^1 \cong S^2$ *thus making* S^3 *a principal fiber bundle over* S^2.

We also denote by $h : S^3 \to S^2$ *the natural projection. The above construction is called the Hopf fibration of* S^3. *Hopf showed that* $[h] \in \pi_3(S^2)$ *is non-trivial i.e.* $[h] \neq id$ *and generates* $\pi_3(S^2)$ *as an infinite cyclic group i.e.* $\pi_3(S^2) \cong \mathbf{Z}$. *This class* $[h]$, *also known as the* Hopf invariant, *corresponds to the Dirac monopole quantization condition (see section 5.1 for the details). The Hopf fibration of* S^3 *can be extended to the unit sphere* $S^{2n-1} \subset \mathbf{C}^n$. *The quotient space in this case is the complex projective space* $\mathbf{CP}^{n-1}$. *This fibration arises in the geometric quantization of the isotropic harmonic oscillator.*

One can similarly consider the quaternionic Hopf fibration as follows. Observe that

$$SU(2) \cong \{\, x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbf{H} \mid |x| = 1 \,\}.$$

$SU(2)$ acts on $\mathbf{H}^n$ on the right by quaternionic multiplication. This action leaves the unit sphere $S^{4n-1} \subset \mathbf{H}^n$ invariant and induces a fibration of S^{4n-1} over the quaternionic projective space $\mathbf{HP}^{n-1}$. For the case $n = 2$, $\mathbf{HP}^1 \cong S^4$ and the Hopf fibration gives S^7 as a non-trivial principal $SU(2)$ bundle over S^4. This bundle plays a fundamental role in our discussion of the BPST instanton.

Bott periodicity

The higher homotopy groups of the classical groups were calculated by R. Bott in the course of proving his well known periodicity theorem. Several of the groups appearing in this theorem have been used in physical theories. We give below a table of the higher homotopy groups of $U(n)$, $SO(n)$ and $SP(n)$ and indicate the stable range in which the periodicity appears.

Stable homotopy of the classical groups

	$U(n), 2n > k$	$SO(n), n > k+1$	$SP(n), 4n > k-2$
π_1	$\mathbf{Z}$	$\mathbf{Z}_2$	0
π_2	0	0	0
π_3	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
π_4	0	0	$\mathbf{Z}_2$
π_5	$\mathbf{Z}$	0	$\mathbf{Z}_2$
π_6	0	0	0
π_7	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
π_8	0	$\mathbf{Z}_2$	0
period	2	8	8

Some homotopy groups outside the stable range also arise in gauge theories. For example, $\pi_3(SO(4)) = \mathbf{Z} \oplus \mathbf{Z}$ is closely related to the self-dual and the anti-self-dual solutions of the Yang-Mills equations on S^4. $\pi_7(SO(8)) = \mathbf{Z} \oplus \mathbf{Z}$ arises in the solution of the Yang-Mills equations on S^8. This solution and similar solutions on higher dimensional spheres satisfy certain generalized duality conditions (see [MA9] for details.)

Classifying spaces

Let G be a Lie group. The classification of principal G-bundles over a manifold M is achieved by the use of classifying spaces. A topological space $B_k(G)$ is said to be *k-classifying for G* if the following conditions hold:

(i) there exists a contractible space $E_k(G)$ on which G acts freely and $B_k(G)$ is the quotient of $E_k(G)$ under this G-action such that

$$E_k(G) \to B_k(G)$$

is a principal fiber bundle;

(ii) given a manifold M of dimension $\leq k$ and a principal bundle $P(M,G)$, there exists a continuous map $f : M \to B_k(G)$ such that the pull-back $f^*(E_k(G))$ to M is bundle-isomorphic to P.

It can be shown that homotopic maps give rise to equivalent bundles and that all principal G-bundles over M arise in this way. Let $[M, B_k(G)]$ denote the set of equivalence classes under homotopy, of maps from M to $B_k(G)$. Then the *classifying property* may be stated as follows.

CLASSIFYING PROPERTY. *There exists a one-to-one correspondence between the set* $[M, B_k(G)]$ *and the set of isomorphism classes of principal* G*-bundles over* M.

The spaces $E_k(G)$ and $B_k(G)$ may be taken to be manifolds for a fixed k. However classifying spaces can be constructed for arbitrary finite dimensional manifolds. They are denoted by $E(G)$ and $B(G)$ and are in general infinite dimensional. The spaces $E(G)$ and $B(G)$ are called *universal classifying spaces* for principal G-bundles.

EXAMPLE 4.4. *The Hopf fibration* $S^{2n+1} \to \mathbb{CP}^n$ *is a principal* $U(1)$*-bundle and is* n*-classifying. By forming a tower of these fibrations by inclusion, i.e. by considering*

$$S^3 \subset S^5 \subset S^7 \subset \cdots$$

$$\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3 \subset \cdots,$$

we obtain the spaces S^∞ *and* $\mathbb{CP}^\infty$ *such that* S^∞ *is a principal* $U(1)$*-fibration of* $\mathbb{CP}^\infty$. *Thus* $B(U(1)) = \mathbb{CP}^\infty$ *and* $E(U(1)) = S^\infty$. *This classification is closely related to the Dirac monopole quantization condition (see Example 5.1).*

EXAMPLE 4.5. *The quaternionic Hopf fibration* $S^{4n+3} \to \mathbb{HP}^n$ *is a principal* $SU(2)$*-bundle and is* n*-classifying. By considering the towers*

$$S^7 \subset S^{11} \subset S^{15} \subset \cdots$$

$$\mathbb{HP}^1 \subset \mathbb{HP}^2 \subset \mathbb{HP}^3 \subset \cdots,$$

we obtain the universal classifying spaces $B(SU(2)) = \mathbb{HP}^\infty$ *and* $E(SU(2)) = S^\infty$. *This example is closely related to the classification of instantons (see section* 6.3 *).*

4.2. Differential complexes and cohomology

A *differential complex* is a finite sequence of vector spaces V_i, $-1 \leq i \leq n+1$, and linear transformations $L_i \in Hom(V_{i-1}, V_i)$, $0 \leq i \leq n+1$ such that

$$L_{i+1} \circ L_i = 0, \qquad 0 \leq i \leq n. \tag{4.1}$$

It is customary to set $V_{-1} = V_{n+1} = \{0\}$ and $L_0 = L_{n+1} = 0$ and to write the complex as follows

$$0 \xrightarrow{L_0} V_0 \xrightarrow{L_1} V_1 \to \cdots \to V_{n-1} \xrightarrow{L_n} V_n \xrightarrow{L_{n+1}} 0.$$

We write

$$V = \bigoplus_{i=-1}^{n+1} V_i$$

and

$$L = \bigoplus_{i=0}^{n+1} L_i$$

and denote the above differential complex by the pair (V, L). We observe that the generalized deRham sequence (see section 3.3) is an example of a sequence which fails to be a complex, the obstruction being given by the curvature.

Given a differential complex (V, L) we define a *cocycle of order i* as an element $\alpha \in ker L_{i+1}$ and we define a *coboundary of order i* as an element $\beta \in Im L_i$. We denote by Z^i (resp. B^i) the set of all cocycles (resp. coboundaries) of order i. In view of the condition (4.1), B^i is a subspace of Z^i. The quotient space Z^i/B^i is denoted by H^i and is called the *i-th cohomology space* of the complex (V, L). We write

$$H^*(V, L) = \bigoplus_{i=0}^{n} H^i$$

and call $H^*(V, L)$ the total cohomology space of (V, L). The operators L_i are frequently referred to as *coboundary operators.*

In differential geometry we are interested in the situation where the vector spaces are spaces of smooth sections of vector bundles and the coboundary operators are the linear differential or pseudo-differential operators on sections. In what follows we consider complex vector bundles; the corresponding theory for real vector bundles can be developed along similar lines. Let E (resp. F) be a complex vector bundle of rank n (resp. l) over a differential manifold M . Let

$$L : \Gamma(E) \to \Gamma(F)$$

be a C-linear operator from the sections of E to the sections of F . L is called a *(linear) differential operator of order* k from E to F if locally L can be expressed as follows

$$L(s) = \sum_{|I|\leq k} a^I \partial_I(s), \qquad s \in \Gamma(E)$$

where

$I = (i_1, i_2, \ldots, i_r)$ is a multi-index, $1 \leq r \leq m$, $m = dim M$,
$|I| = i_1 + i_2 + \cdots + i_r$ is called the length of I ,
$\partial_I = \partial^{|I|}/\partial x_1^{i_1} \cdots \partial x_r^{i_r}$ in the coordinate neighborhood U in consideration,
a^I is a function on U such that $a^I(x) \in L(E_x, F_x)$, $\forall x \in U$.

Alternatively we can express this definition by saying that L factors through the k-jet extension $J^k(E)$ of E , i.e. there exists a vector bundle morphism $f : J^k(E) \to F$ such that $L = f_* \circ j^k$, where $f_* : \Gamma(J^k(E)) \to \Gamma(F)$ is the map induced by f or that the following diagram commutes

$$\begin{array}{ccc} & \Gamma(E) & \\ {}^{j^k}\swarrow & & \searrow^{L} \\ \Gamma(J^k(E)) & \underset{f_*}{\longrightarrow} & \Gamma(F) \end{array}$$

In fact, this formulation can easily be extended to define a non-linear differential operator of order k between sections of arbitrary fiber bundles. We denote by $D_k(E,F)$ the set of all linear differential operators of order k from E to F . In order to study the properties of differential operators, we now define the k-symbol of an operator $L \in D_k(E,F)$. Let

$$T_0^*M = T^*M \setminus \{ \text{the image of the zero section} \}$$

and let $p_0 : T_0^*M \to M$ be the restriction of the canonical projection $p : T^*M \to M$ to T_0^*M . Let p_0^*E be the pull-back of the bundle E to the space T_0^*M so that

$$\begin{array}{ccc} p_0^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_E} \\ T_0^*M & \underset{p_0}{\longrightarrow} & M \end{array}$$

We define the *k-symbol* of L, $\sigma_k(L) \in Hom(p_0^* E, p_0^* F)$ by

$$\sigma_k(L)(\alpha_x, e) = L(t)(x)$$

where $(\alpha_x, e) \in p_0^* E$ and $t \in \Gamma(E)$ is defined as follows. We can choose $s \in \Gamma(E)$ and $f \in \mathcal{F}(M)$ such that $s(x) = e$ and $df(x) = \alpha_x$; then

$$t = \frac{i^k}{k!}(f - f(x))^k s.$$

It can be shown that $L(t)(x)$ depends only on α_x and e and is independent of the various local choices made. The set of k-symbols is the set $Smbl_k(E, F)$ defined by

$$Smbl_k(E,F) := \{\sigma \in Hom(p_0^* E, p_0^* F) \mid \sigma(c\alpha_x, e) = c^k \sigma(\alpha_x, e), \\ c > 0, \alpha_x \in T_0^* M\}.$$

Given $L \in D_k(E, F)$ there exists the formal adjoint L^* of L such that $L^* \in D_k(F, E)$. We have

$$\sigma_k(L^*) = (\sigma_k(L))^*.$$

We say that $L \in D_k(E, F)$ is *elliptic* if $\sigma_k(L)$ is an isomorphism. In particular $rankE = rankF$. Sobolev spaces play a fundamental role in the study of differential operators and in particular, in the study of operators arising in gauge theories. For an introduction to the theory of Sobolev spaces, see for example, R. Adams [bAD1].

THEOREM 4.1. *Let $L \in D_k(E, F)$ be an elliptic operator and L^* its formal adjoint. Then*

(i) *L^* is elliptic*

(ii) *L is Fredholm (i.e. $kerL$ and $cokerL$ are finite dimensional) on a suitable (Hilbert) Sobolev space $H_s \subset \Gamma(E)$.* ■

For an elliptic operator $L \in D_k(E, F)$ we define the *index*, $Ind(L)$ of L, by

$$Ind(L) = dim\ kerL - dim\ cokerL = dim\ kerL - dim\ kerL^*.$$

Let (V, L) be a differential complex where $V_i = \Gamma(E_i)$, E_i is a vector bundle of rank n_i over M and L_i is a linear differential operator from E_{i-1} to E_i . (V, L) is said to be *elliptic* if the associated symbol sequence

$$0 \to p_0^*(E_0) \stackrel{\sigma(L_1)}{\to} p_0^*(E_1) \stackrel{\sigma(L_2)}{\to} \cdots \stackrel{\sigma(L_n)}{\to} p_0^*(E_n) \to 0$$

is exact. Given the elliptic complex (E, L) we define the *j-th Laplacian operator* Δ_j of the complex by

$$\Delta_j = L_{j+1}^* L_{j+1} + L_j L_j^* \; : \; \Gamma(E_j) \to \Gamma(E_j), \; 0 \leq j \leq n.$$

The operator Δ_j is a self-adjoint (i.e. $\Delta_j = \Delta_j^*$) elliptic operator, $\forall j$. We denote by $Z^q(E, L)$, $B^q(E, L)$, $H^q(E, L)$ respectively the cocycles, coboundaries, cohomology spaces of order q of the complex (E, L). The homogeneous solutions of $\Delta_j(s) = 0$ are called the *harmonic* sections. The space of all harmonic sections is a finite-dimensional subspace of $\Gamma(E_i)$ which is isomorphic to the j-th cohomology space $H^j(E, L)$. This statement is in fact the content of the Hodge theory when (E, L) is the deRham complex of M. The *index* of the elliptic complex is defined by

$$Ind(E, L) = \sum_{k=0}^{n} (-1)^k dim H^k(E, L).$$

It turns out that this *analytic* index can be expressed in term of *topological* invariants (characteristic classes) associated to the complex (E, L). This is the content of the classical Atiyah-Singer index theorem.

The deRham cohomology of a manifold M is defined as follows. Let $T_{\mathbb{C}}^* M = T^* M \otimes \mathbb{C}$ be the complexified cotangent bundle. We denote by $\Lambda_{\mathbb{C}}^i(M)$ the space of sections $\Lambda^i(T_{\mathbb{C}}^* M)$. $\Lambda_{\mathbb{C}}^i(M)$ is the space of $\mathbb{C}$-valued differential forms on M of degree i. The exterior differential operator d extends to $\Lambda_{\mathbb{C}}(M) = \oplus_{i=0}^m \Lambda_{\mathbb{C}}^i(M)$ and is also denoted by d. The *deRham complex* is the following

$$0 = \Lambda_{\mathbb{C}}^{-1} \xrightarrow{0} \Lambda_{\mathbb{C}}^0 \xrightarrow{d} \Lambda_{\mathbb{C}}^1 \cdots \xrightarrow{d} \Lambda_{\mathbb{C}}^n \xrightarrow{0} \Lambda_{\mathbb{C}}^{n+1} = 0.$$

d is a differential operator of order 1 and its symbol is given by

$$\sigma_1(d)(\alpha_x, \beta) = i\alpha \wedge \beta, \qquad \beta \in \Lambda_{\mathbb{C}}^j(M), \forall j.$$

It is easy to verify that $d \circ d = 0$ and that the symbol sequence is exact. Thus the deRham complex $(\Lambda_{\mathbb{C}}, d)$ is an elliptic complex. The Laplacian

$$\Delta_k = d \circ \delta + \delta \circ d : \Lambda_{\mathbb{C}}^k(M) \to \Lambda_{\mathbb{C}}^k(M)$$

is called the *Hodge Laplacian* on k-forms, where δ is the codifferential operator. If M is a Riemannian manifold then Δ_0, the Hodge Laplacian on functions (0-forms), coincides with the classical Laplace-Beltrami operator. The solutions of $\Delta_k \alpha = 0$, $\alpha \in \Lambda_{\mathbb{C}}^k(M)$, were called by Hodge *harmonic* k-forms. The *Hodge decomposition theorem* for closed k-forms can be expressed by the following formula

$$\alpha = d\beta + \delta\gamma + \theta$$

where α is a closed k-form, $\beta \in \Lambda^{k-1}$, $\gamma \in \Lambda^{k+1}$, θ is a harmonic k-form. For α in a given cohomology class, the harmonic form θ is uniquely determined. Thus we have an isomorphism of the k-th cohomology space H^k with the space of harmonic k-forms. The k-th *Betti number* b_k is defined by $b_k = dim H^k$. Thus the *index* of the deRham complex (Λ, d) can be written as

$$Ind(\Lambda, d) = \sum_{k=0}^{n}(-1)^k b_k.$$

It is well known that the number on the right-hand side is the Euler characteristic $\chi(M)$. Thus we have a topological characterization of the analytic index $Ind(\Lambda, d)$. This may be regarded as a very special case of the index theorem.

EXAMPLE 4.6. *A single elliptic operator may be considered as an elliptic complex. We give the example of the Hirzebruch signature operator* D^+ *and state the relation of its index to the Hirzebruch signature of* M. *Let* M *be a compact, oriented Riemannian manifold of dimension* $4n$. *We define the involution operator* $j : \Lambda^k \to \Lambda^{4n-k}$ *by*

$$\alpha \mapsto j(\alpha) = i^{k(k-1)+2n} * \alpha = (-1)^n i^{k(k-1)} * \alpha.$$

The operator j *extends to* Λ *and satisfies* $j^2 = 1$. *We denote by* Λ_+ *(resp.* Λ_-*) the eigenspace of* j *for the eigenvalue* $+1$ *(resp.* -1*). Define* $D^+ = d + \delta|_{\Lambda_+}$. *Then*

$$D^+ : \Lambda_+ \to \Lambda_-$$

is an elliptic operator called the Hirzebruch signature operator. *We now define the Hirzebruch signature* $Sign(M)$ *of* M. *Consider the bilinear operator*

$$h : H^{2n} \times H^{2n} \to R$$

defined by

$$(\alpha, \beta) \mapsto \int_M (\alpha \wedge \beta).$$

In the above formula we have used the same letter to denote the cohomology class and a $2n$*-form representing that class. It can be shown that* h *is a symmetric non-degenerate form. Let* (e^+, e^-) *be the signature of the bilinear form* h. *Then the* Hirzebruch signature *is defined by*

$$Sign(M) = e^+ - e^-.$$

The Hirzebruch signature theorem *shows that the index of the signature operator equals the Hirzebruch signature of M i.e.*

$$Ind(D^+) = Sign(M).$$

The Hirzebruch signature theorem is a special case of the Atiyah-Singer index theorem. A brief discussion of the index theorem is given at the end of this chapter.

If M is 4-dimensional, then the bilinear form h defined in the above example induces a bilinear form

$$h : H^2(M,\mathbf{Z}) \times H^2(M,\mathbf{Z}) \to \mathbf{Z}$$

This symmetric, non-degenerate form is called the *intersection form* of M. The definition given above works only for smooth manifolds, however, the intersection form is defined also for topological manifolds and plays a fundamental role in the study of 4-manifolds. A brief discussion of its significance is given in section 6.4.

The discussion of homotopy and cohomology given in sections 4.1 and 4.2 forms a small part of an area of mathematics called Algebraic Topology, where these and other related concepts are developed for general topological spaces. Two standard references for this material are W.S. Massey [bMA2], E.H. Spanier [bSP1]. A very readable introduction is given in F.H. Croom [bCR1].

4.3. Characteristic classes

Let $P(M,G)$ be a principal bundle over a manifold M with structure group G. We will define a set of cohomology classes in $H^*(M,\mathbf{R})$ (the cohomology ring of M with real coefficients) associated to the principle bundle, which characterize the bundle up to bundle isomorphisms. They are called the characteristic classes of $P(M,G)$. There are several different ways of defining these characteristic classes (D. Husemoller [bHU1], J.W. Milnor, J.D. Stasheff [bMI2]). We define them by using connections in the bundle P and the Weil homomorphism defined below.

Let $\rho : G \to GL(V)$ be a representation of G on the real vector space V. Let $S^k(V)$ denote the set of k-linear symmetric functions on V and let $S(V) = \oplus_{k=0}^{\infty} S^k(V)$ be the symmetric algebra of V. The group G acts on $S^k(V)$ by an action induced by the representation ρ, which we also denote by ρ. We define

$$I^k(V,\rho) := \{f \in S^k(V) \mid \rho(a)(f) = f,\ \forall a \in G\}.$$

The space $I^k(V,\rho)$ is called the space of *G-invariant symmetric polynomials of degree k on V*, the action of G by ρ being understood. We define the space $I(V,\rho)$

of all G-invariant symmetric polynomials on V by

$$I(V,\rho) := \bigoplus_{k=0}^{\infty} I^k(V,\rho).$$

The space $I(V,\rho)$ becomes an algebra with multiplication induced by the symmetric product $\vee$ on $S(V)$. In what follows we will take V to be the Lie algebra $\mathfrak{g}$ of the Lie group G and ρ to be the adjoint representation ad of G on $\mathfrak{g}$ and we will denote $I^k(V,\rho)$ by $I^k(G)$ and $I(V,\rho)$ by $I(G)$.

Let ω be the connection 1-form of a connection on $P(M,G)$ and Ω the curvature 2-form of ω. For $f \in I^k(G)$, we define the $2k$-form f_Ω on P by

$$f_\Omega(X_1,\ldots,X_{2k}) = \frac{1}{(2k)!}\sum_{\sigma\in S_{2k}} sign(\sigma) f(u_1,\ldots,u_k),$$

where

$$u_i = \Omega(X_{\sigma(2i-1)}, X_{\sigma(2i)}) \in \mathfrak{g}, \quad 1 \le i \le k$$

and S_{2k} is the symmetric group of permutations on $2k$ numbers. Since the curvature form Ω is tensorial with respect to the adjoint action of G on $\mathfrak{g}$ and f is G-invariant, the $2k$-form f_Ω on P descends to a $2k$-form $\hat{f}_\Omega$ on M, i.e. there exists a form $\hat{f}_\Omega \in \Lambda^{2k}(M)$ such that

$$\pi^*(\hat{f}_\Omega) = f_\Omega,$$

where π is the canonical bundle projection of P on M. It can be shown that the form $\hat{f}_\Omega$ is closed and hence defines a cohomology class $[\hat{f}_\Omega] \in H^{2k}(M,\mathbb{R})$. Furthermore, this cohomology class turns out to be independent of the particular connection ω on P, i.e. if ω_1,ω_2 are two connections on P with respective curvature forms Ω_1,Ω_2 then $[\hat{f}_{\Omega_1}] = [\hat{f}_{\Omega_2}] \in H^{2k}(M,\mathbb{R})$. In view of this result we can define a map

$$w_k : I^k(G) \to H^{2k}(M,\mathbb{R}) \quad \text{by} \quad w_k(f) := [\hat{f}_\Omega].$$

The family of maps $\{w_k\}$ defines the map

$$w : I(G) \to H^*(M,\mathbb{R})$$

which is called the *Weil homomorphism*. We note that w is an algebra homomorphism of the algebra $I(G)$ into the cohomology algebra $H^*(M,\mathbb{R})$.

The construction discussed above extends to the algebra $I_{\mathbb{C}}(G)$ of the complex valued G-invariant polynomials on $\mathbf{g}$ to give us the complex algebra homomorphism

$$w_{\mathbb{C}} : I_{\mathbb{C}}(G) \to H^*(M, \mathbb{C}).$$

This homomorphism is called the *Chern-Weil homomorphism.*

An element $p \in w(I(G)) \subset H^*(M, \mathbf{R})$ is called a *real characteristic class* of the bundle P. Similarly, an element $c \in w_{\mathbb{C}}(I_{\mathbb{C}}(G)) \subset H^*(M, \mathbb{C})$ is called a *complex characteristic class* of the bundle P.

Characteristic classes are topological invariants of the principal bundle P and characterize P up to isomorphism. They can also be viewed as topological invariants of the vector bundle associated to P by the fundamental or defining representation of the structure group G. It is possible to give an axiomatic formulation of characteristic classes of vector bundles directly (S. Kobayashi, K. Nomizu [bKO2], J.W. Milnor, J.D. Stasheff [bMI2]). In studying the properties of characteristic classes we will use either of these formulations as convenient.

For the Lie groups that are commonly encountered in physical applications we can show that the algebra of characteristic classes is finitely generated and exhibit a basis for the algebra. Before giving several examples of this, we make the following remark, which simplifies our work. Given a function $p_k(A)$ which is a homogeneous polynomial of degree k in the entries of the matrix A, we can define a symmetric polynomial p in k variables by a process of polarization. For example, if $p_2(A)$ is homogeneous of degree 2, we can define a symmetric function p of 2 variables by the formula

$$p(A_1, A_2) = \frac{1}{2}(p_2(A_1 + A_2) - p_2(A_1) - p_1(A_2)).$$

Similar formulas can be given for higher order polynomials. In view of this remark, in the examples discussed below, we give only the homogeneous polynomials $p_k(A)$ for the construction of the characteristic classes.

EXAMPLE 4.7. $G = GL(n, \mathbf{R})$ *; then* $I(G)$ *is generated by* $p_0, p_1, \ldots, p_n$ *where the* p_i *'s are defined by*

$$det(\lambda I_n - \frac{1}{2\pi}A) = \sum_{i=0}^{n} p_i \lambda^{n-i},$$

$A \in \mathbf{gl}(n, \mathbf{R})$. *The* $4k$*-form corresponding to* p_{2k} *determines the* k-th Pontryagin class *of* P. *If* Ω *is the curvature form of some connection* ω *of* P *and* $\hat{\Omega}$ *the corresponding* 2*-form on* M, *then we can write a* $4k$*-form* $p_{2k}(\Omega)$ *representing the* k*-th Pontryagin class as follows*

$$p_{2k}(\Omega) = \frac{1}{(2\pi)^{2k}(2k)!}\delta^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}\hat{\Omega}^{i_1}_{j_1} \wedge \cdots \wedge \hat{\Omega}^{i_{2k}}_{j_{2k}}$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ *and the* $\hat{\Omega}^i_j$ *'s are the components of* $\hat{\Omega}$ *in* $\mathbf{gl}(n, \mathbf{R})$.

EXAMPLE 4.8. $G = O(n, \mathbf{R})$. *The characteristic algebra is generated by* $p_0, p_2, p_4, \ldots$ *defined as in Example 4.7. Observe that for* $A \in o(n, \mathbf{R}) = so(n, \mathbf{R})$ *one has*

$$det(\lambda I_n - \frac{1}{2\pi}A) = det(\lambda I_n + \frac{1}{2\pi}A).$$

Therefore $p_1 = p_3 = \cdots = 0$ *and* p_{2k} *again corresponds to the* k*-th Pontryagin class. In Example 4.7,* $p_{2k+1} \neq 0$ *in general but* $w(p_{2k+1}) = 0$, *since every* $GL(n, \mathbf{R})$ *-connection is reducible to an* $O(n, \mathbf{R})$ *-connection.*

EXAMPLE 4.9. *With the notation of Example 4.8, we construct an* $SO(2m)$ *-invariant polynomial, called the Pfaffian, which is not invariant under the action of* $O(2m)$. *The* Pfaffian $Pf(\hat{\Omega})$ *is defined by*

$$Pf(\hat{\Omega}) = \frac{1}{2^{2m}\pi^m m!}\sum_{\sigma} sgn(\sigma)\hat{\Omega}_{\sigma(1)\sigma(2)}\cdots\hat{\Omega}_{\sigma(2m-1)\sigma(2m)}.$$

The class $w(Pf)$ *is called the* Euler class *of* $P(M, SO(2m))$, *which occurs as the generalized curvature in the Chern-Gauss-Bonnet theorem which states that*

$$\int_M Pf(\hat{\Omega}) = \chi(M).$$

This theorem generalizes the classical Gauss-Bonnet theorem for compact surfaces in $\mathbf{R}^3$. *For such a surface the* $Pf(\hat{\Omega})$ *is a multiple of the volume form on* M . *The multiplier* k *is the Gaussian curvature and the above theorem reduces to*

$$\int_M k = \chi(M).$$

The expression for the Euler class of a 4-manifold as an integral of a polynomial in the Riemann curvature was obtained by Lanczos [LA2] *but he did not recognize its topological significance. The general formula for the Euler class of an arbitrary oriented manifold was obtained by Chern and it was in studying this generalization that Chern was led to his famous characteristic classes, namely the Chern classes.*

EXAMPLE 4.10. $G = GL(n, \mathbb{C})$. $I_{\mathbb{C}}(G)$ *is generated by the characteristic classes* $c_0, c_1, \ldots, c_n$ *defined by*

$$det(\lambda I_n - \frac{1}{2\pi i}A) = \sum_{i=0}^{n} c_i\lambda^{n-i}.$$

The classes $w(c_k) \in H^{2k}(M, \mathbb{C})$, $k = 1, 2, \ldots, n$, *are called the* Chern classes *of* P .

EXAMPLE 4.11. *$G = U(n)$. $I_{\mathbb{C}}(G)$ is generated by the classes c_k defined in Example 4.10. However for $A \in u(n)$ one has the following complex conjugation condition:*

$$det(\lambda I_n - \frac{1}{2\pi i}A) = \overline{det(\lambda I_n - \frac{1}{2\pi i}A)}.$$

Therefore the class $w(c_k)$ turns out to be a real cohomology class, i.e. $w(c_k) \in H^{2k}(M,\mathbf{R}) \subset H^{2k}(M,\mathbb{C})$, where $H^{2k}(M,\mathbf{R})$ is regarded as a subset of $H^{2k}(M,\mathbb{C})$ by the isomorphism induced by the inclusion of $\mathbf{R}$ into $\mathbb{C}$. In view of the fact that the $GL(n,\mathbb{C})$-connection is reducible to a $U(n)$-connection we find that the classes $w(c_k)$ of Example 4.10 are in fact real.

As we have indicated above the characteristic classes turn out to be real cohomology classes. Indeed the normalizing factors, that we have used in defining them, make them integral cohomology classes.

The *total Chern class* $c(P)$ of P is defined by

$$c(P) := 1 + c_1(P) + \cdots + c_n(P),$$

and the *Chern polynomial* $c(t)$ is defined by

$$c(t) := \sum_{i=0}^{n} t^{n-i} c_i(P) = t^n + t^{n-1} c_1(P) + \cdots + t c_{n-1}(P) + c_n(P).$$

We factorize the Chern polynomial formally as follows

$$c(t) = (t + x_1)(t + x_2) \cdots (t + x_n).$$

The Chern classes are then expressed in terms of the formal generators $x_1, \ldots, x_n$ by elementary symmetric polynomials. For example

$$c_1(P) = x_1 + x_2 + \cdots + x_n$$

$$c_2(P) = \sum_{i<j} x_i x_j$$

$$\vdots \qquad \vdots$$

$$c_n(P) = x_1 x_2 \cdots x_n.$$

The *Chern character* $ch(P)$ is defined by

$$ch(P) = \sum_{i=1}^{n} \exp(x_i) \in H^*(M,\mathbf{Q}),$$

where the exp on the right-hand side is interpreted as a formal series which in fact is finite and terminates after $\frac{1}{2}(dim\ M)$ terms. The first few terms on the right-hand side, expressed in term of the Chern classes, give us the following formula

$$ch(P) = n + c_1(P) + [\frac{1}{2}c_1^2(P) - c_2(P)] + \cdots .$$

Another important class is the *Todd class* $\tau(P) \in H^*(M, \mathbf{Q})$ defined by

$$\tau(P) = \prod_{i=1}^{n} \frac{x_i}{1 - \exp(-x_i)}.$$

The characteristic classes satisfy the following properties.
(i) Naturality with respect to pull back of bundles:

If $f : N \to M$ and f^*P is the pullback of the principal bundle $P(M, G)$ to N, then we have

$$b(f^*(P)) = f^*(b(P)),$$

where $b(P)$ is a characteristic class of the bundle P.
(ii) The Whitney sum rule:

$$b(E_1 \oplus E_2) = b(E_1)b(E_2),$$

where E_1, E_2 are vector bundles and b is a characteristic class.

If E is a complex vector bundle with fiber $\mathbb{C}^n$ which is associated to $P(M, GL(n, \mathbb{C}))$ then the k-th Chern class $c_k(E)$ of E is represented by $w(c_k) \in H^{2k}(M, \mathbb{C})$. If F is a real vector bundle over M and F^c its complexification, then the k-th Pontryagin class $p_k(F)$ of F is given by

$$p_k(F) = (-1)^k c_{2k}(F^c) \in H^{4k}(M, \mathbb{R}).$$

We work out in detail the Chern classes for an $SU(2)$-bundle over a compact manifold. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a matrix in the Lie algebra $su(2)$. The algebra of characteristic classes is generated by the images of the invariant symmetric polynomials $TrA = a_{11} + a_{22}$ and

$det A = a_{11}a_{22} - a_{12}a_{21}$. Writing $a_{lk} = -\Omega_{lk}/2\pi i$ we get the polynomials in curvature representing the Chern classes

$$c_1(P) = -\frac{1}{2\pi i}(\Omega_{11} + \Omega_{22}) = 0 \qquad \text{(by the Lie algebra property)}$$

$$c_2(P) = -\frac{1}{4\pi^2}(\Omega_{11} \wedge \Omega_{22} - \Omega_{12} \wedge \Omega_{21}) = \frac{1}{8\pi^2}Tr(\Omega \wedge \Omega).$$

The 2-form $\hat{\Omega}$ induced by the curvature on the base will be denoted by F. We will see later that F corresponds to a gauge field. Evaluating the second Chern class on the fundamental cycle of M, we get

$$c_2(P)[M] = \frac{1}{8\pi^2}\int_M Tr(F \wedge F).$$

In view of the integrality of the Chern classes, the number k defined by

$$k = -c_2(P)[M]$$

is an integer which is called the *topological charge* or *topological quantum number* of the principal $SU(2)$-bundle. The topological charge may also be defined as $p_1(P)[M]$, where p_1 is the first Pontryagin class. If M is orientable and $dim\ M = 4$, then we have

$$k = p_1(P) = -c_2(P).$$

The index theorem

The Atiyah-Singer index theorem is one of the most important results of modern mathematics. A good introduction to this result may be found in R. Palais [bPA2] and P. Shanahan [bSH1]. Its application to gauge theories is discussed in the book by B. Booss and D. Bleecker [bBO1]. We give below a statement of one version of the index theorem and explain its relation to some of the special cases discussed in this chapter.

THEOREM. *(Atiyah-Singer) Let M be a compact manifold of dimension m and let (E, L) be an elliptic differential complex over M. Then there exists a compactly supported cohomology class $a(E, L) \in H_c^*(TM, \mathbf{Q})$ such that*

$$Ind(E, L) = (a(E, L).td(TM \otimes \mathbb{C}))([TM]),$$

where td is the Todd class and $[TM]$ is the fundamental class of the tangent bundle.

The statemment of the index theorem takes a much simpler form in the special case when all the Laplacians of the elliptic complex are second order operators. In this case it can be shown that there exists a cohomology class $b(E, L) \in H^m(M)$ with the property that $b(E, L) = 0$ for odd m and

$$Ind(E, L) = (b(E, L))([M]).$$

In particular, m odd implies that $ind(E, L) = 0$. The index theorems for classical elliptic complexes are special cases of this formula as indicated below.
1) For the DeRham complex $E_i = \Lambda^i_{\mathbb{C}}$, $L = d$, $b(E, L) =$ Euler class of M and the index theorem takes the form $Ind(E, L) = \chi(M)$, the Euler characteristic of M.
2) For the signature complex of $4k$-dimensional manifold M, $E_\pm = \Lambda_\pm$, $L = D$, $b(E, L) =$ Hirzebruch's L_k-polynomial (a polynomial in the Pontryagin classes of M) and the index theorem takes the form $Ind(E, L) = Sign(M)$, the Hirzebruch signature of M.
Similar formulation can be given for the Dolbeault complex of a complex manifold and the spin complex of a spin manifold.

5. GAUGE FIELDS AND THEIR ASSOCIATED FIELDS

5.1. Pure gauge fields

The theory of gauge fields and their associated fields, such as the Yang-Mills-Higgs fields, was developed by physicists to explain and unify the fundamental forces of nature. The theory of connections in a principal fiber bundle was developed by mathematicians during approximately the same period, but the fact that they are closely related was not noticed for many years. Since then substantial progress has been made in understanding this relationship and in applying it successfully to problems in both physics and mathematics. Standard references for a general discussion of gauge theories are the books by D. Bleecker [bBL], B. Booss, D. Bleecker [bBO1], L. Boutet de Monvel, A. Douady, J.-L. Verdier, eds. [bBO2], W. Drechsler, M. Mayer [bDR1] and the papers [DA1], [EG1].

In physical applications one is usually interested in a fixed Lie group G called the *gauge group* which represents an internal or local symmetry of the field. The base manifold M of the principal bundle $P(M, G)$ is usually the space-time manifold or its Euclidean version, i.e. a Riemannian manifold of dimension 4. But in some physical applications, such as superspace, Kaluza-Klein and string theories, the base manifold can be an essentially arbitrary manifold.

Let M be an m-dimensional manifold and $P(M, G)$ a principal bundle with the gauge group G as its structural group. A connection in P is called a *gauge connection*. The connection 1-form ω is called the *gauge connection form* or simply the *gauge connection*. A *global gauge* or simply a *gauge* is a section $s \in \Gamma(P)$. The gauge potential

A on M in gauge s is given by $A = s^*(\omega)$. A global gauge and hence the gauge potential on M exists if and only if the bundle P is trivial. A *local gauge*, is a section of the bundle $P(M, G)$ restricted to some open subset $U \subset M$. Local gauges defined for the local representations $(U_i, \psi_i)_{i \in I}$ of P, always exist. Let $t \in \Gamma(U_i, P)$ be a local gauge; then the 1-form $t^*\omega \in \Lambda^1(U_i, \mathbf{g})$ is called the *gauge potential* in the local gauge t and is denoted by A_t. If the local gauge t is given we often denote A_t by A and call it a *local gauge potential.* Let $\Omega = d^\omega \omega$ be the curvature 2-form of ω with values in the Lie algebra $\mathbf{g}$. We call Ω the *gauge field on* P. Although this terminology is fairly standard, we would like to warn the reader that sometimes, in the physics literature, our gauge potential is called the gauge field and our gauge field is called the field strength tensor. As we have seen in section 3.1 above there exists a unique 2-form F_ω on M with values in the Lie algebra bundle $ad\, P$ associated to the curvature 2 -form Ω such that

$$F_\omega = s_\Omega,$$

The 2-form $F_\omega \in \Lambda^2(M, ad\, P)$ is called the *gauge field on* M corresponding to the gauge connection ω. The gauge field F_ω is globally defined on M, even though, in general, there is no corresponding globally defined gauge potential on M. If we are given a local gauge potential $A_t \in \Lambda^1(U_i, \mathbf{g})$, then on U_i we have the relations

$$t^*\omega = A_t \text{ and } F_\omega = d^\omega A_t$$

EXAMPLE 5.1. (The Dirac Monopole) *Let $S^3(S^2, U(1))$ be the principal $U(1)$-bundle over S^2 determined by the Hopf fibration of S^3. Let μ denote the connection 1-form of the canonical connection on this bundle and let F_μ be the corresponding gauge field on S^2. In this case there is no globally defined gauge potential on S^2. We need at least two charts to cover S^2 and therefore, at least two local potentials which give rise to a single globally defined gauge field. This field can be shown to be equivalent to the Dirac monopole field. The Dirac monopole quantization condition corresponds to the classification of principal $U(1)$-bundles over S^2. These are classified by $\pi_1(U(1)) \cong \mathbf{Z}$. In general, the principal G-bundles over S^2 are classified by $\pi_1(G)$. Thus $\pi_1(SU(2)) = id$ implies that there is a unique $SU(2)$-monopole on S^2 and $\pi_1(SO(3)) = \mathbf{Z}_2$ implies that there are two inequivalent $SO(3)$-monopoles on S^2 (see* [GR1], [WU4], [WU5], [YA4], [YA6] *for further details).*

We denote by $\mathcal{A}(P)$ the *space of gauge connections* on P defined by

(5.1a) $$\mathcal{A}(P) := \{\, \omega \in \Lambda^1(P, \mathbf{g}) \mid \omega \text{ is a connection on } P \,\}.$$

If P is fixed we will denote $\mathcal{A}(P)$ simply by $\mathcal{A}$ and a similar notation will be followed for other related spaces. The space $\mathcal{A}$ is an affine space of the underlying vector space $\Lambda^1(M, ad\,P)$. Indeed from the condition (3.1) it follows that $\omega_1, \omega_2 \in \mathcal{A}$ implies that $\omega_1 - \omega_2$ is horizontal and therefore defines a unique 1-form on M with values in the bundle $ad\,P$. Then we have that, for a fixed connection α

$$\mathcal{A} \cong \{\alpha + \pi^* A \mid A \in \Lambda^1(M, ad\,P)\}. \tag{5.1b}$$

Thus the tangent space $T_\alpha \mathcal{A}$ is isomorphic to $\Lambda^1(M, ad\,P)$ and we identify the two spaces. If $\langle\, ,\, \rangle_{\mathbf{g}}$ is an invariant inner product on $\mathbf{g}$,we have a natural inner product defined on $T_\alpha \mathcal{A}$ as follows

$$\langle A, B\rangle_\alpha = \int_M g^{ij} \langle A_i, B_j\rangle_{\mathbf{g}}, \quad \forall A, B \in T_\alpha \mathcal{A} \tag{5.2}$$

where g^{ij} are the components of the metric tensor g on M with respect to a base dx^i of $T_x^* M$ and $A = A_i dx^i$, $B = B_i dx^i$. We obs erve that an invariant inner product always exists for semisimple Lie algebras and is given by a multiple of the Killing form K on $\mathbf{g}$ given by

$$K(x, y) = -Tr(ad_x, ad_y).$$

The inner product defined in (5.2) can be easily extended to $\Lambda^k(M, ad\,P)$. A connection ω on P defines a covariant derivative

$$\nabla^\omega : \Lambda^0(M, ad\,P) \to \Lambda^1(M, ad\,P)$$

which is compatible with the metric on M, i.e.

$$\langle \nabla^\omega_X \psi,\ \phi\rangle + \langle \nabla^\omega_X \phi,\ \psi\rangle = X(\langle \phi,\ \psi\rangle),$$
$$\forall \phi,\ \psi \in \Lambda^0(M, ad\,P),\ X \in \mathcal{X}(M).$$

If λ is the Levi-Civita connection on (M, g), then we can define the covariant derivative

$$\nabla^{(\omega,\lambda)} : \Lambda^k(M, ad\,P) \to \Gamma(T^*M \otimes A^k T^*M \otimes ad\,P)$$

or simply ∇^ω by

$$\nabla^\omega \equiv \nabla^{(\omega,\lambda)} := 1 \otimes \nabla^\omega + \nabla^\lambda \otimes 1.$$

The corresponding covariant exterior derivative is denoted by d^ω and its adjoint is denoted by δ^ω.

Gauge potentials and gauge fields acquire physical significance only after one postulates the field equations to be satisfied by them. These equations and their consequences must then be subjected to suitably devised experiments for verification. On more than one occasion a theory was abandoned, when its predictions seemed to contradict an experimental result and later this experiment or its conclusions turned out to be incorrect and the abandoned theory turned out to be correct. In any case there is no natural mathematical method for assigning field equations to gauge fields. Thus the Riemann curvature of a space-time manifold M is the gauge field corresponding to the gauge potential given by the Levi-Civita connection on the orthonormal frame bundle of M, but it does not describe the gravitational field until it is subjected to Einstein's field equations. If instead it satisfies Yang-Mills equations, then it describes a classical Yang-Mills field. This aspect of gauge fields is already evident in the following remark of Yang [YA2]: «The electromagnetic field is a gauge field. Einstein's gravitational theory is intimately related to the concept of gauge fields, although to *identify* the gravitational field as a gauge field is not an absolutely straightforward matter.» However, a study of physically interesting field equations such as Maxwell's equations of electro-magnetic field and their quantization indicates some desirable features for the gauge field equations. One of these features is gauge invariance of the field equations. This requirement is formulated in terms of the group of gauge transformations which acts on the various fields involved. In the following paragraph we give a mathematical formulation of this group.

The group $Diff(P)$ of the diffeomorphisms of P is too large to serve as a group of gauge transformations, since it mixes up the fibers of P. The requirement that fibers map to fibers may be expressed by the condition that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[f_M]{} & M \end{array}$$

When this condition is satisfied, we say that $f \in Diff(P)$ covers $f_M \in Diff(M)$. The set $Diff_M(P)$ of all $f \in Diff(P)$ such that f covers some $f_M \in Diff(M)$, is a group called the *group of generalized gauge transformations*. We note that the fiber preserving property of f completely determines the diffeomorphism f_M. We define the group $\mathcal{G}$ to be the subgroup $Aut(P) \subset Diff_M(P)$ of principal bundle automorphisms of P. Thus

$$(5.3) \qquad \mathcal{G} := Aut(P) = \{f \in Diff_M(P) \mid f_M = id_M\}.$$

Then $\mathcal{G}$ is a normal subgroup of $Diff_M(P)$. We call $\mathcal{G}$ the *group of gauge transformations*. From the definition (5.3) it is clear that the group $\mathcal{G}$ maps each fiber of P

into itself and we have the following exact sequence of groups

$$1 \to \mathcal{G} \xrightarrow{i} Diff_M(P) \xrightarrow{j} Diff(M) \to 1,$$

where i denotes the inclusion map and j is defined by

$$j(f) = f_M, \quad \forall f \in Diff_M(P).$$

In several applications one is interested in splitting the above exact sequence or in constructing an extension of $Diff(M)$ by $\mathcal{G}$. In particular, we often want to lift the action of $Diff(M)$ to some subgroup of $Diff_M(P)$. Additional geometric structures may also be involved in this process. For example, if $P = L(M)$, the bundle of frames of M, then it is a principal bundle but carries the additional structure given by the soldering form θ (see section 3.1) and we have the following proposition.

PROPOSITION 5.1. *There exists a natural lift* $\lambda : Diff(M) \to Diff(L(M))$ *which splits the exact sequence of groups*

$$1 \to \mathcal{G} \xrightarrow{i} Diff_M(L(M)) \xrightarrow{j} Diff(M) \to 1.$$

Furthermore, $f \in Diff(L(M))$ *is the natural lift of a diffeomorphism* $f_M \in Diff(M)$ *(i.e* $f = \lambda(f)$ *) if and only if* f *leaves the soldering form invariant, i.e* $f^*\theta = \theta$. ∎

When M is a four dimensional Lorentz manifold, connections on the frame bundle $L(M)$ play the role of gravitational potentials. Action functionals involving connections and metrics on M form the starting point of gauge theories of gravitation.

The group $\mathcal{G}$ acts on the space of gauge connections $\mathcal{A}(P)$ by pulling back the connection form. i.e.

$$(f, \omega) \mapsto f \cdot \omega = f^*\omega, \quad f \in \mathcal{G}, \omega \in \mathcal{A}(P).$$

We say that connections $\alpha, \beta \in \mathcal{A}$ are *gauge equivalent* if there exists a gauge transformation $f \in \mathcal{G}$ such that $\beta = f \cdot \alpha$. From the definition of the action of $\mathcal{G}$ on $\mathcal{A}$ given above, it follows that each equivalence class of gauge eq uivalent connections is an orbit of $\mathcal{G}$ in $\mathcal{A}$. The orbit space $\mathcal{O} = \mathcal{A}/\mathcal{G}$ thus represents gauge inequivalent connections and is called the *moduli space of gauge connections* on $P(M, G)$.

A physical interpretation of a gauge transformation $f \in \mathcal{G}$ is that f is a local (pointwise) change of gauge over M. For this reason it is sometimes called a local gauge group, but we will not use this terminology. There are several alternative definitions of the group of gauge transformations. We have collected together the most frequently used definitions in the following theorem.

THEOREM 5.1. *There exists a one-to-one correspondence between each pair of the following three sets:*

(i) *the group of gauge transformations* $\mathcal{G}$,

(ii) *the set* $\mathcal{F}_G(P,G)$ *of all functions* $f : P \to G$ *such that* f *is* G*-equivariant, with respect to the adjoint action of* G *on itself,*

(iii) *the set* $\Gamma(P \times_{Ad} G)$. *of sections of the associated bundle* $P \times_{Ad} G$ *over* M.

Proof. The correspondence between $\mathcal{F}_G(P,G)$ and $\Gamma(P \times_{Ad} G)$ is a special case of the correspondence between $\mathcal{F}_G(P,F)$ and $\Gamma(E(M,F,r,P))$ with $F = G$ and $r = Ad$, the adjoint action of G on itself (see section 2.2). For $g \in \mathcal{G}$ we define $f_g : P \to G$ by

$$f_g(u) = a,$$

where $a \in G$ is the unique element such that $g(u) = ua$. It can be verified that $g \longmapsto f_g$ is a one-to-one correspondence from $\mathcal{G}$ to $\mathcal{F}_G(P,G)$ with inverse given by the map from $\mathcal{F}_G(P,G)$ to $\mathcal{G}$ such that $f \longmapsto g_f$ where $g_f(u) = uf(u)$. ∎

In view of this theorem we use any one of the three representations above for the group of gauge transformations as needed. For example, regarding $\mathcal{G}$ as the space of sections of $(P \times_{Ad} G)$, the bundle of groups (not a principal G-bundle), we can show that a suitable Sobolev completion of $\mathcal{G}$ (also denoted by $\mathcal{G}$) is a Banach Lie group (i.e., $\mathcal{G}$ is a Banach manifold with smooth group operations). Let ad denote the adjoint action of the Lie group G on its Lie algebra $\mathbf{g}$. Let $E(M, \mathbf{g}, ad, P)$ be the associated vector bundle with fiber type $\mathbf{g}$ and action ad, the adjoint action of G on $\mathbf{g}$. Recall that this bundle is a bundle of Lie algebras denoted by $P \times_{ad} \mathbf{g}$ or adP. We denote $\Gamma(ad\,P)$ by $\mathcal{LG}$; it is a Lie algebra under pointwise bracket and pointwise exponential map to $\mathcal{G}$. We will show in the next section that a suitable Sobolev completion of $\mathcal{LG}$ is a Banach Lie algebra which is the Lie algebra of the infinite dimensional Banach Lie group $\mathcal{G}$.

5.2. The space of gauge connections

Without any assumption of compactness for M or G it can be shown that $\mathcal{G}$ is a Schwartz Lie group (i.e. a Lie group modeled on a Schwartz space) with Lie algebra consisting of sections of $ad\ P$ of compact support. A discussion of this approach is developed in [AB1], [AB2], [MI1]. While this approach has the advantage of working in full generality, the technical difficulties of working with spaces modeled on an arbitrary locally convex vector space can be avoided by considering Sobolev completions of the relevant objects as follows.

In this section we consider a fixed principal bundle $P(M,G)$ with compact connected oriented Riemannian base manifold M and compact gauge group G. These assumptions are satisfied by most Euclidean gauge theories of physical interest. The base manifold is typically a sphere S^n or a torus T^n or their products such as $S^n \times T^m$. Thus for $n = 4$ one frequently considers as a base $S^4, T^4, S^3 \times S^1, S^2 \times S^2$. With appropriate boundary conditions on gauge fields one may also include non-compact bases such as $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$. The gauge group G is generally one of the following: $U(n)$, $SU(n)$, $O(n)$, $SO(n)$ or one of their products. For example the gauge group of unified electroweak theory is $SU(2) \times U(1)$. As we discussed above the gauge connections (gauge potentials) and the gauge fields acquire physical significance only after field equations, to be satisfied by them, are postulated. However the topology and geometry of the space of gauge connections has significance for all physical theories and especially for the problem of quantization of gauge theories. Various aspects of the topology and geometry of the space of gauge connections and its orbit spaces have been studied in [AR1], [AT9], [BA2], [BO2], [CO3], [CO4], [GA2], [KO1], [MI3], [MO2], [SI6], [WA1], [YA3].

Let E be a Riemannian vector bundle over a compact Riemannian manifold M associated to $P(M,G)$. Recall that $\Lambda^p(M,E) = \Gamma(A^p(M) \otimes E)$ is the space of p-forms on M with values in E. Fixing a connection α on P and using the induced covariant derivative ∇^α on $\Lambda^p(M,E)$, we define a Sobolev k-form on $\Lambda^p(M,E)$ by

$$||\phi||_k = (\sum_{j=0}^{k} \int_M |(\nabla^\alpha)^j \phi|^2)^{1/2}, \qquad \phi \in \Lambda^p(M,E).$$

The completion of $\Lambda^p(M,E)$ in this norm is a Hilbert space under the associated bilinear form denoted by $H_k(\Lambda^p(M,E))$ .A different choice of the connection on P and metrics on M and E gives an equivalent norm. The map $d^\alpha : \Lambda^p(M,E) \to \Lambda^{p+1}(M,E)$ extends to a smooth bounded map of Hilbert spaces (also denoted by d^α)

$$d^\alpha : H_k(\Lambda^p(M,E)) \to H_{k-1}(\Lambda^{p+1}(M,E)).$$

For $p = 0$ this map has finite dimensional kernel and closed range. In general the sequence

$$0 \to H_k(\Lambda^0(M,E)) \xrightarrow{d^\alpha} H_{k-1}(\Lambda^1(M,E)) \xrightarrow{d^\alpha} \cdots$$

fails to be a complex, the obstruction being provided by the curvature of α. In particular fixing a connection α on P gives an identification of $\mathcal{A}$ with $\Lambda^1(M, ad\, P)$ and we denote the corresponding Sobolev completion of $\mathcal{A}$ in the k-norm by $H_k(\mathcal{A})$.

The curvature map $\omega \to \Omega = d^\omega \omega$ extends to a smooth bounded Hilbert space map from $H_{k+1}(\mathcal{A})$ into $H_k(\Lambda^2(M, ad\,P))$ for $k \geq 1$. The Sobolev completion of the gauge group $\mathcal{G}$ is obtained by considering $\mathcal{G}$ as a subset of $\Lambda^0(M, P \times_\rho End(V))$, where $\rho : G \to End(V)$ is a faithful representation of the gauge group G. We define $H_k(\mathcal{G})$ to be the closure of $\mathcal{G}$ in $H_k(\Lambda^0(M, P \times_\rho End(V)))$. The Lie algebra structure of $\Lambda^1(M, ad\,P)$ extends to a Lie algebra structure on its Sobolev completion $H_k(\Lambda^1(M, ad\,P))$ and we have the following theorem.

THEOREM 5.2. *For* $k > \frac{1}{2}(dim M + 1)$, $H_k(\mathcal{G})$ *is an infinite dimensional Lie group modeled on a separable Hilbert space with Lie algebra* $H_k(\Lambda^0(M, ad\,P))$. *The action of* $\mathcal{G}$ *on* $\mathcal{A}$ *defined in section 5.1 extends to a smooth action of* $H_k(\mathcal{G})$ *on* $H_{k-1}(\mathcal{A})$. ∎

From now on we consider that all objects requiring Sobolev completions have been completed in appropriate norms and drop the H_k from $H_k(object)$.

In many applications one is interested in the orbit space $\mathcal{O} = \alpha/\mathcal{G}$ whose points correspond to gauge equivalent connections. The orbit space is given the quotient topology and is a Hausdorff topological space. However, in general, the action of $\mathcal{G}$ on $\mathcal{A}$ is not free and $\mathcal{O}$ fails to be a manifold. We now discuss two methods of suitably modifying $\mathcal{A}$ or $\mathcal{G}$ to obtain orbit spaces with nice mathematical structure.

1. Let $\mathcal{G}_0 \subset \mathcal{G}$ denote the group of *based gauge transformations*, defined by

$$\mathcal{G}_0 = Aut_0 P = \{ f \in Aut P \mid f(u) = u \quad \text{for some fixed } u \in P\}.$$

A based gauge transformation that fixes a connection is the identity. Therefore $\mathcal{G}_0$ acts freely on $\mathcal{A}$ and the orbit space $\mathcal{O}_0 = \mathcal{A}/\mathcal{G}_0$ is an infinite dimensional Hilbert manifold. $\mathcal{A}(\mathcal{O}_0, \mathcal{G}_0)$ is a principal fiber bundle with canonical projection

$$\pi_0 : \mathcal{A} \to \mathcal{A}/\mathcal{G}_0 = \mathcal{O}_0.$$

2. Let $Z(G)$ denote the centre of the gauge group G. Denote by $\mathcal{Z}$ $\Gamma(P \times_{Ad} Z(G)) \cong Z(\mathcal{G})$. Let $\mathcal{G}_c = \mathcal{G}/\mathcal{Z}$. Let $\mathcal{A}_{ir} \subset \mathcal{A}$ denote the space of irreducible connections. Then $\mathcal{G}_c$ acts freely on $\mathcal{A}_{ir}$ and we denote by $\mathcal{O}_{ir}$ the orbit space $\mathcal{A}_{ir}/\mathcal{G}_c$. $\mathcal{O}_{ir}$ is an infinite dimensional Hilbert manifold and $\mathcal{A}_{ir}(\mathcal{O}_{ir}, \mathcal{G}_c)$ is a principal fiber bundle with canonical projection

$$\pi_c : \mathcal{A}_{ir} \to \mathcal{A}_{ir}/\mathcal{G}_c = \mathcal{O}_{ir}.$$

Now we use this construction to study the Gribov ambiguity.

Gribov ambiguity

The group of gauge transformations $\mathcal{G}$ acts on $\mathcal{A}$. We denote this action by $(g,\omega) \mapsto g\omega$, $g \in \mathcal{G}$, $\omega \in \mathcal{A}$. The induced action on the curvature Ω is given by $g.\Omega = g\Omega g^{-1}$. We denote by $\mathcal{O}$ the orbit space of gauge equivalent connections (gauge potentials), i.e.

$$\mathcal{O} = \mathcal{A}/\mathcal{G}.$$

We denote by

$$p : \mathcal{A} \to \mathcal{O} = \mathcal{A}/\mathcal{G}$$

the natural projection. In the Feynman integral approach to quantization one must integrate over this orbit space to avoid the infinite contribution coming from gauge equivalent fields. However, the mathematical nature of this space is essentially unknown. Physicists often try to get around this difficulty by choosing a section $s : \mathcal{O} \to \mathcal{A}$ and integrating over its image $s(\mathcal{O}) \subset \mathcal{A}$ with a suitable weight factor such as the Fadeev-Popov determinant, which may be thought of as the Jacobian of the change of variables effected by $p_{|s(\mathcal{O})} : s(\mathcal{O}) \to \mathcal{O}$. This procedure amounts to a choice of one connection in $\mathcal{A}$ from each equivalence class in $(\mathcal{O})$ and is referred to as *gauge fixing.* The question of the existence of such sections is thus crucial for this approach. For the trivial $SU(2)$ bundle over $\mathbf{R}^4$, Gribov showed that the so called Coulomb gauge fails to be a section, i.e. the Coulomb gauge is not a true gauge globally. Gribov showed that the local section corresponding to the Coulomb gauge at the 0 connection if extended (under some boundary conditions) intersects the orbit through 0 at large distances and thus fails to be a section. The boundary conditions imposed by Gribov amount to the gauge potential being defined over the compactification of $\mathbf{R}^4$ to S^4. He also discussed a similar problem for $\mathbf{R}^3$. This non-existence of a global gauge is referred to as the *Gribov ambiguity*. In view of this negative result, it is natural to ask if any true gauge exists under these boundary conditions. Without any boundary conditions it is possible to exhibit a global gauge, but it does not seem to have any physical meaning. We show that in fact the Gribov ambiguity is present in all physically relevant cases, so that no global gauge exists. For further details see [CH2], [GR3], [GR4], [JA1], [SI5].

The Gribov ambiguity is a consequence of the topology of the configuration space as we now explain. Let $P(S^4, SU(2))$ be a principal bundle. Recall that $\mathcal{A}$ is isomorphic to the vector space of 1-forms on S^4 with values in the vector bundle $adP = P \times_{ad} \mathfrak{g}$ once a connection is fixed. If $\alpha \in \mathcal{A}$ is a fixed connection we can write

$$\mathcal{A} \cong \{ \alpha + A \mid A \in \Lambda^1(S^4, adP) \}.$$

Consider the slice S_α defined by

$$S_\alpha := \{ \alpha + A \mid \delta^\alpha A = 0 \} \subset \mathcal{A}.$$

In particular if $\alpha = 0$ then

$$S_0 = \{ A \mid \delta^0 A = 0 \}.$$

We call this the generalized Coulomb gauge. Locally the condition $\delta^0 A = 0$ can be written as

$$A^i_{,i} = \sum_i \frac{\partial A^i}{\partial x^i} = 0 .$$

Locally (or on $\mathbb{R}^4$ as a base) one can find a connection gauge equivalent to the given connection with zero time component. The gauge condition then reduces to the classical Coulomb gauge

$$div\, A \;=\; 0 .$$

It is convenient to reformulate the definition of the group $\mathcal{G}$ of gauge transformations as follows. Let

$$E_2 = E(M, \mathbb{C}^2, r, P)$$

be the vector bundle associated to P with fiber $\mathbb{C}^2$, where r is the defining or standard representation of $SU(2)$ on $\mathbb{C}^2$. Then

$$\mathcal{G} \cong \{ h \in \Gamma(Hom(E_2, E_2)) \mid h(x) \in SU(2) \ , \forall x \in M \}.$$

Define the isotropy group $\mathcal{G}_\alpha$ of a fixed connection α by

$$\mathcal{G}_\alpha \;=\; \{ g \in \mathcal{G} \mid g \cdot \alpha = \alpha \}.$$

It is easy to see that $g \in \mathcal{G}_\alpha$ if and only if

$$d^\alpha g \;=\; 0 .$$

In particular g is completely determined by specifying its value at a single point, say, $x_0 \in M$. To study the question of the irreducibility of α, we consider the holonomy group H_α of the connection α at x_0. We observe that $g \in \mathcal{G}_\alpha$ if and only if $g(x_0) \in SU(2)$ and $[g(x_0), H_\alpha] = 0$. If α is irreducible then this is equivalent to requiring that $g(x_0) \in Z(SU(2))$ (the center of $SU(2)$ which is isomorphic to $\mathbf{Z}_2$). We define the group $\mathcal{G}_c$ as the quotient of $\mathcal{G}$ by the center $Z(G)$ of G, which in this case gives

$$\mathcal{G}_c = \mathcal{G}/\mathbf{Z}_2 .$$

Let $\mathcal{A}_{ir} \subset \mathcal{A}$ be the set of the irreducible connections. $\mathcal{G}_c$ acts on $\mathcal{A}_{ir}$ and the quotient $\mathcal{O}_{ir}$ is the orbit space of irreducible connections. $\mathcal{A}_{ir}$ is a principle $\mathcal{G}_c$-bundle over $\mathcal{O}_{ir}$, i.e.

$$\mathcal{A}_{ir} = P(\mathcal{O}_{ir}, \mathcal{G}_c).$$

Let $f : S^k \to \mathcal{A}_{ir} \subset \mathcal{A}$ be a continuous map. Regarding f as a map of the boundary of a $(k+1)$-simplex Δ_{k+1} we can extend f linearly to a map of Δ_{k+1} to $\mathcal{A}$. It can be shown that the extended map is homotopic to a map g which actually lies in $\mathcal{A}_{ir}$. The construction uses the fact that the set $\mathcal{A}_r$ of reducible connections is a closed nowhere dense stratified subset of $\mathcal{A}$. A simple argument then shows that $f \sim g \sim c_\alpha$, where c_α is the constant map $c_\alpha(x) = \alpha$, $\forall x \in S^k$. Thus

(5.4) $$\pi_k(\mathcal{A}_{ir}) = id .$$

Under certain topological conditions it can be shown that $\mathcal{A}_{ir} = \mathcal{A}$. For example if $P(M, SU(2))$ is a non trivial bundle and for the second cohomology space we have $H^2(M, \mathbf{Z}) = 0$, then $\mathcal{A}_{ir} = \mathcal{A}$. In particular this condition is satisfied by $M = S^4$.

We now show that for a fixed non-trivial $SU(2)$-bundle over S^4 , there exists some k such that

(5.5) $$\pi_k(\mathcal{G}_c) \neq id .$$

Assuming the result (5.5) we can prove that no gauge $s : \mathcal{O} \to \mathcal{A}$ exists in this case. For if such an s exists then

$$s_{|\mathcal{O}_{ir}} : \mathcal{O}_{ir} \to \mathcal{A}_{ir}$$

is a cross section of the principal $\mathcal{G}_c$-bundle $\mathcal{A}_{ir}$ and we have a corresponding trivialization $\mathcal{A}_{ir} = \mathcal{O}_{ir} \times \mathcal{G}_c$. Therefore, for some k,

$$0 = \pi_k(\mathcal{A}_{ir}) = \pi_k(\mathcal{O}_{ir}) \times \pi_k(\mathcal{G}_c) \neq 0 .$$

To complete our argument we now sketch a proof of (5.5) . By definition of $\mathcal{G}_c$ we have the following exact sequence

$$0 \to \mathbf{Z}_2 \to \mathcal{G} \to \mathcal{G}_c \to 0 .$$

Therefore if $\mathcal{G}$ is connected then $\pi_1(\mathcal{G}_c)$ is non-trivial. For $k > 1$ we have $\pi_k(\mathcal{G}) = \pi_k(\mathcal{G}_c)$. Recall that the group $\mathcal{G}_0$ is a group of based gauge transformations over some fixed point of the manifold, which we may take to be the point at infinity (i.e. the north pole) on S^4 . We have the following short exact sequence

$$0 \to \mathcal{G}_0 \to \mathcal{G} \to SU(2) \to 0.$$

It induces the following long exact sequence in homotopy

$$\cdots \to \pi_{k+1}(SU(2)) \to \pi_k(\mathcal{G}_0) \to \pi_k(\mathcal{G}) \to \pi_k(SU(2)) \to \cdots.$$

In particular we have

$$\begin{aligned} &\cdots \to \pi_3(\mathcal{G}) \to \pi_3(SU(2)) \to \pi_2(\mathcal{G}_0) \to \pi_2(\mathcal{G}) \to \\ &\to \pi_2(SU(2)) \to \cdots. \end{aligned} \tag{5.6}$$

In the present case of an $SU(2)$-bundle over S^4 it can be shown that

$$\pi_k(\mathcal{G}_0) \cong \pi_{k+4}(SU(2)).$$

In particular $\pi_2(\mathcal{G}_0) \cong \pi_6(SU(2)) = \mathbf{Z}_{12}$. We also know that $\pi_3(SU(2)) = \mathbf{Z}$ and $\pi_2(SU(2)) = 0$. Thus (5.6) becomes

$$\cdots \to \pi_3(\mathcal{G}) \to \mathbf{Z} \to \mathbf{Z}_{12} \to \pi_2(\mathcal{G}) \to 0.$$

Thus if $\pi_2(\mathcal{G}) = 0$ then $\pi_3(\mathcal{G}) \neq 0$. Thus either $\pi_2(\mathcal{G}) \neq 0$ or $\pi_3(\mathcal{G}) \neq 0$.

The non-existence of a global gauge fixing need not prevent an application of path-integral methods. For example one may use the fact that the orbit space $\mathcal{O}_{ir}$ and the space $\mathcal{A}_{ir}$ are paracompact. Thus we may be able to find a suitable locally finite covering and a subordinate partition of unity for $\mathcal{O}_{ir}$ and construct local gauges and local weights to define the path integrals. However for this procedure to work we need an explicit description of the locally finite covering and local weights for the Fadeev-Popov approach.

5.3. Associated fields and coupled equations

Let $E(M, F, \tau, P)$ be a vector bundle associated to the principal bundle $P(M, G)$. We call a section $\phi \in \Gamma(E)$ an *E-potential* (or a generalized Higgs potential) and $d^\omega \phi$ an *E-field* (or a generalized Higgs field) associated to the gauge connection ω . If $E = ad(P) := P \times_{ad} \mathfrak{g}$ then ϕ is called the *Higgs potential* and $d^\omega \phi$ the *Higgs field* associated to the gauge connection ω . We call $ad(P)$ the *Higgs bundle.* In general there are several fields that can be defined on bundles associated with a given manifold.

For example, on a Lorentz 4-manifold the Levi-Civita connection is interpreted as representing a gravitational potential. Recall that the Levi-Civita connection is the unique torsion-free connection defined on the bundle of orthonormal frames $\mathcal{O}(M)$ of M. In general if M is a pseudo-Riemannian manifold, we can define the space of linear connections $\mathcal{A}(\mathcal{O}(M))$ on M by

$$\mathcal{A}(\mathcal{O}(M)) :=$$
$$= \{ \alpha \in \Lambda^1(\mathcal{O}(M), so(m)) \mid \alpha \text{ is a connection on } \mathcal{O}(M) \}.$$

Generalized theories of gravitation often use this $\mathcal{A}(\mathcal{O}(M))$ as their configuration space. If M is a spin manifold and $S(M)(M, Spin(m))$ the $Spin(m)$-principal bundle one can consider the space $\mathcal{A}(S(M))$ of spin connections on the bundle $S(M)$, defined by

$$\mathcal{A}(S(M)) :=$$
$$= \{ \beta \in \Lambda^1(S(M), spin(m)) \mid \beta \text{ is a connection on } S(M) \}.$$

For a given signature (p, q) we may consider the space $\mathcal{RM}_{(p,q)}(M)$ of all pseudo-Riemannian metrics on M of signature (p, q). The space $\mathcal{RM}_{(m,0)}(M)$ of Riemannian metrics on M is denoted simply by $\mathcal{RM}(M)$. Recall that there is a canonical principal $GL(m, \mathbf{R})$-bundle over M, namely $L(M)$ the bundle of frames of M. If $\rho : GL(m, \mathbf{R}) \to End\ V$ is a representation of $GL(m, \mathbf{R})$ on V and $E(M, V, \rho, L(M))$ is the corresponding associated bundle of $L(M)$, then we denote by $\mathcal{W}$ the space of E-potentials $\Gamma(E)$, i.e.

$$\mathcal{W} = \Gamma(E(M, V, \rho, L(M))).$$

Thus we see that we have an array of fields on a given base manifold M and we must specify the equations governing the evolution and interactions of these fields and study their physical meaning. There is no standard procedure for doing these things. In many physical applications one obtains the coupled field equations of interacting fields as the Euler-Lagrange equations of a variational problem with the Lagrangian constructed from the fields. For any given problem the Lagrangian is chosen subject to certain invariance or covariance requirements related to the symmetries of the fields involved. We now discuss three general conditions for Lagrangians that are frequently imposed in physical theories. In what follows we restrict ourselves to a fixed, compact 4-manifold M as the base manifold, but the discussion can be easily extended to apply to an arbitrary base manifold. Let $P(M, G)$ be a principal bundle over M whose structure group G carries a bi-invariant metric h. For example if G is a semisimple Lie group, then a suitable multiple of the Killing form on $\mathbf{g}$ provides a bi-invariant metric on G.

We want to consider coupled field equations for a metric $g \in \mathcal{RM}(M)$, a potential $\phi \in \mathcal{W}(M)$ (a section of the bundle associated to the frame bundle $L(M)$), a connection $\omega \in \mathcal{A}(P)$ and a generalized Higgs potential $\psi \in \mathcal{H} = \Gamma(E(M, V_r, r, P))$, where $r : G \to End(V_r)$ is a representation of the gauge group G . Thus our *configuration space* is defined by

$$\mathcal{C} := \mathcal{RM} \times \mathcal{W} \times \mathcal{A} \times \mathcal{H}.$$

We assume that the field equations are the variational equations of an action integral defined by a Lagrangian L on the configuration space with values in $\Lambda^4(M)$. When a fixed volume form such as the metric volume form is given we regard L as a real valued function. We shall use any one of these conventions without comment. The action $\mathcal{E}$ is given by

$$\mathcal{E}(g, \phi, \omega, \psi) = \int_M L(g, \phi, \omega, \psi).$$

There are various groups associated with the geometric structures involved in the construction of these fields, which have natural actions on them. We note the following exact sequence of groups:

$$0 \to \mathcal{G} \to Diff_M(P) \to Diff(M).$$

We shall require the Lagrangian to satisfy the following conditions:
i) Naturality, ii) Local regularity, iii) Conformal invariance.

i) *Naturality* : Naturality with respect to the group of generalized gauge transformations is defined as follows. Let $F \in Diff_M(P)$ be a generalized transformation covering $f \in Diff(M)$. Then by naturality with respect to $Diff_M(P)$ we mean that

$$L(f^*g, f^*_\rho\phi, F^*\omega, F^*_r\psi) = f^*L(g, \phi, \omega, \psi), \tag{5.7}$$

where f^*_ρ is the induced action of f on $\mathcal{W}$, and F^*_r is the action induced by the generalized gauge transformation F on $\mathcal{H}$. In the absence of the principal bundle P this condition reduces to naturality with respect to $Diff(M)$ and is Einstein's condition of general covariance of physical laws derived from the Lagrangian formalism. Further in the absence of $\mathcal{W}(M)$ this condition corresponds to the covariance of gravitational field equations when the Lagrangian is taken to be the standard Einstein-Hilbert Lagrangian. If we require naturality with respect to the group $\mathcal{G}$ then the condition (5.7) is precisely the principle of gauge invariance introduced by Weyl. Since in this case $f = id$ the condition (5.7) becomes

$$L(g, \phi, F^*\omega, F^*_r\psi) = L(g, \phi, \omega, \psi).$$

The concept of natural tensors on a Riemannian manifold was introduced in [EP1] and was extended to oriented Riemannian manifolds in [ST3] where a functorial formulation of naturality is given and a complete classification of natural tensor fields is given under some regularity conditions.

ii) *Local regularity* : Given any coordinate chart on M and a local gauge we can express the various potentials and fields with respect to induced bases. We require that in this system the Lagrangian be expressible as a universal polynomial in

$$(det\ g)^{-1/2},\ (det\ h)^{-1/2},\ g_{ij},\ \partial^{|\alpha|}g_{ij}/\partial x^{\alpha},\ \phi_{|\beta|},\ \omega_{|\gamma|},\ \psi_{|\delta|}\ \cdots$$

where $\alpha,\ \beta,\ \gamma,\ \delta\cdots$ are suitable multi-indices (i.e. in the coefficients and derivatives of the potentials and fields in the induced bases). In physical applications one often restricts the order of derivatives that can occur to 2. For example in gravitation one considers natural tensors satisfying the conditions that they contain derivatives up to order 2 and depend linearly on the second order derivatives. Then it is well known that such tensors can be expressed as

$$c_1 R^{ij} + c_2 g^{ij} S + c_3 g^{ij},$$

where R^{ij} are the components of the Ricci tensor and S is the scalar curvature. Einstein's equations with or without the cosmological constant involve the above combination with suitable values of the constants $c_1,\ c_2,\ c_3$. Applying the classification theorem of [ST3] to $SO(4)$ actions on the metric and gauge fields, we get the following general form for the Lagrangian:

$$\begin{aligned} L(g,\ \omega) = {} & c_1 S^2 + c_2||K||^2 + c_3||W^+||^2 + \\ & + c_4||W^-||^2 + c_5||F_\omega \wedge F_\omega|| + c_6||F_\omega \wedge (*F_\omega)||, \end{aligned} \tag{5.8}$$

where $S,\ K,\ W^+,\ W^-$ are the $SO(4)$-invariant components of the Riemannian curvature (see section 3.2), and F_ω is the gauge field of the gauge potential ω. For a suitable choice of constants in the above Lagrangian we obtain various topological invariants of M and P as well as the pure Yang-Mills action. For example the first Pontryagin class of M is given by

$$p_1(M) = \frac{1}{4\pi^2}\int_M (||W^+||^2 - ||W^-||^2).$$

The first Pontryagin class of P is given by

$$p_1(P) = \frac{1}{8\pi^2}\int_M (||F_\omega^+||^2 - ||F_\omega^-||^2),$$

which turns out to be the instanton number of P (see section 6.2).

To satisfy the conditions of naturality and local regularity for fields coupled to gauge fields physicists often start with ordinary derivatives of associated fields and the coupling is achieved by replacing these by gauge covariant derivatives in the Lagrangian. This is called the principle of minimal coupling (or interaction). An interesting geometrical discussion of this may be found in [PE1]. These two requirements can also be formulated by taking the Lagrangian to be defined on sections of suitable jet bundles on the space of connections. Using this approach a generalization of the classical theorem of Utiyama [UT1] has been obtained in [GA2] and [MA3].

iii) *Conformal invariance* : The condition of conformal invariance of the Lagrangian may be expressed as follows

$$L(e^{2\sigma}g,\ \phi,\ \omega,\ \psi) = L(g,\ \phi,\ \omega,\ \psi),\quad \forall \sigma \in \mathcal{F}(M).$$

In general, Lagrangians satisfying the conditions of naturality and regularity need not satisfy the condition of conformal invariance. This condition is often used to select parameters such as the dimension of the base space and the rank of the representation. A particular case of (5.8) is the Yang-Mills Lagrangian with action

$$\mathcal{E} = \frac{1}{8\pi^2}\int_M F_\omega \wedge *F_\omega .$$

It is an example of a Lagrangian that is conformally invariant only if the dimension of M is 4.

A large number of Lagrangians satisfying the naturality and regularity requirements are used in the physics literature. They are broadly classified into bosonic Lagrangians and fermionic Lagrangians, depending on the absence or presence of spin structures and their associated fields. For example a bosonic Lagrangian is given by

$$L_{boson}(g,\ \omega,\ \psi) = ||F_\omega||^2 + ||\nabla\psi||^2 + \frac{1}{6}S||\psi||^2 - V(\psi),$$

where V is the potential function which is taken to be a gauge invariant polynomial of degree ≤ 4 on the fibers of E. If M is a spin manifold and if Σ is a bundle associated to the spin bundle, then we define an E-valued spinor to be a section of $\Sigma \otimes E$. The fermion Lagrangian is defined by

$$L_{fermion}(g,\ \omega,\ \xi) = ||F_\omega||^2 + \langle \mathcal{D}(\xi),\ \xi\rangle,$$

where $\xi \in \Gamma(\Sigma \otimes E)$ and $\mathcal{D}$ is the Dirac operator on E-valued spinors (see section 4.2). Several important properties of coupled field equations are studied in [PA1], [AR2]. Further references for this and related material are [BO4], [HA4], [LE1].

Removable singularities theorems

Gauge fields and their associated fields arise naturally in the study of physical fields and their interactions. The solutions of these equations are often obtained locally. The question of whether finite energy solutions of the coupled field equations can be obtained globally is of great significance for both the physical and mathematical aspects. The early solutions of $SU(2)$ Yang-Mills field equations in Euclidean setting had a finite number of point singularities when expressed as solutions on the base manifold $\mathbb{R}^4$. The fundamental work of Uhlenbeck ([UH1]-[UH4]) showed that these point singularities in gauge fields are removable by suitable gauge transformations and that these solutions can be extended from $\mathbb{R}^4$ to its compactification S^4 as singularity free solutions of finite energy. These theorems were extended to finite energy solutions of coupled field equations on 4-dimensional Riemannian base manifolds in [PA1]. It was shown in [UH1] that the removable singularities theorems fail in dimensions greater than 4. In dimension 3, the removable nature of isolated singularities is discussed for the Yang-Mills and Yang-Mills-Higgs equations in A. Jaffe, C. Taubles [bJA1] and in [SI1].

An interesting solution of $SU(2)$ Yang-Mills equations over $\mathbb{R}^4$ with fractional topological charge and finite action was obtained in [FO1]. This solution has a set of singular points which constitute a 2-dimensional sphere S^2. As pointed out in [FO2] this solution corresponds to a gauge connection in a principal bundle over $\mathbb{R}^4 \setminus S^2$ and is not a connection over $\mathbb{R}^4$. Indeed this solution can be extended to $S^4 \setminus S^2$ but not to all of S^4. The fractional topological charge arises essentially due to the fact that $\mathbb{R}^4 \setminus S^2$ is not simply connected. It can be shown (see [TA1], [TA2], [SI2], [SI3]) that local Sobolev connections on $SU(2)$-bundles over $\mathbb{R}^4 \setminus M$ (M a smoothly embedded compact 2-manifold in $\mathbb{R}^4$) with finite Yang-Mills action satisfy a certain holonomy condition. If the singular set has codimension greater than 2 then the techniques used for point singularities can be applied to remove these singularities. Thus a codimension 2 singular set, such as an S^2 embedded in $\mathbb{R}^4$, provides an appropriate setting for new techniques and results. For example, the holonomy condition implies that there exist flat connections in a principal bundle over $\mathbb{R}^4 \setminus S^2$ which cannot be extended to a neighborood of the singular set S^2 even though the bundle itself may be topologically trivial. It turns out that a family of such connections may be topologically trivial. It turns out that a family of such connections may be used to obtain a non-trivial connection which can be extended to a neighborhood of the singular set ([CH1], [SI4]). These ideas together with the work on monopoles in hyperbolic 3-space $\mathcal{H}^3$ discussed in [AT4], [BR1] has recently been used in [SI4] to prove the existence of non-dual (and hence non-minimal) solutions of Yang-Mills equations.

6. YANG-MILLS FIELDS

6.1. Yang-Mills fields

The Yang-Mills fields form a special class of gauge fields that have been extensively investigated. In addition to the references given for gauge fields we give the following references which deal with Yang-Mills fields. They are [BO5]-[BO7], [bDO1], [GU1], [M03], [ST2], [UH4], [WE2], [WI2], [WI3], [YA1]-[YA5]. A source-free electromagnetic field is the prototype of Yang-Mills fields. We will show that a source-free electromagnetic field is a gauge field with gauge group $U(1)$. Let $P(M^4, U(1))$ be a principal $U(1)$-bundle over the Minkowski space M^4. Any principal bundle over M^4 is trivializable. We choose a fixed trivialization and use it to write $P(M^4, U(1)) = M^4 \times U(1)$. The Lie algebra $u(1)$ of $U(1)$ may be identified with $i\mathbb{R}$. Thus a connection form on P may be written as $i\omega$, $\omega \in \Lambda^1(P)$, by choosing i as the basis of the Lie algebra $i\mathbb{R}$. The gauge field can be written as $i\Omega$, where $\Omega = d\omega \in \Lambda^2(P)$. The bundle $ad(P)$ is also trivial and we have $ad(P) = M^4 \times u(1)$. Thus the gauge field on the base M^4 can be written as iF, $F \in \Lambda^2(M^4)$. Using the global gauge $s : M^4 \to P$ defined by $s(x) = (x, 1)$, $\forall x \in M^4$ we can pull the connection form $i\omega$ to M^4 to obtain the gauge potential $iA = is^*\omega$. Thus $A \in \Lambda^1(M^4)$ and $F = dA$. The field equations $\delta F = 0$ are obtained as the Euler-Lagrange equations minimizing the action $\int ||F||^2$, where $||F||$ is the pseudo-norm induced by the Lorentz metric on M^4. A gauge transformation f is a section of $Ad(P) = M^4 \times U(1)$. It is completely determined by the function $\psi \in \mathcal{F}(M^4)$ such that

$$f(x) = (x,\ e^{i\psi(x)}) \in Ad(P), \quad \forall x \in M^4.$$

If iB denotes the potential obtained by the action of the gauge transformation f on iA, then we have

$$iB = e^{-i\psi}(iA)e^{i\psi} + e^{-i\psi}de^{i\psi}, \quad \text{or} \quad B = A + d\psi,$$

which is the classical formulation of the gauge transformation f. We observe that the group $\mathcal{G}$ of gauge transformations acts transitively on the space of gauge connections $\mathcal{A}$ and hence the moduli space $\mathcal{A}/\mathcal{G}$ reduces to a single point. This observation plays a fundamental role in the path integral approach to QED (quantum electro-dynamics).

The above considerations can be applied to any simply connected manifold M. In particular, if M is a compact, simply connected, Riemannian manifold and the electromagnetic field $F \in \Lambda^2(M)$ is defined as above, then the equations $dF = 0$, $\delta F = 0$ satisfied by F imply that F is a harmonic 2-form, i.e. F is a solution of the equation

$$\Delta_2 F = 0, \quad \text{where} \quad \Delta_2 = d\delta + \delta d$$

is the Hodge Laplacian on 2-forms. Thus we see that the Euclidean version of Maxwell's equations are precisely the equations for harmonic 2-forms of Hodge theory. An application of Hodge theory now leads to the following

THEOREM 6.1. *Let $P(M, U(1))$ be a principal bundle over a compact, simply connected, Riemannian manifold M. Then the gauge field (curvature) of any gauge potential is the unique harmonic 2-form representing the first Chern class $c_1(P)$.* ∎

Recall that the first Chern class classifies these principal $U(1)$-bundles and is an integral class. When applied to the base manifold S^2 this classification corresponds to the Dirac quantization condition for a monopole (Example 5.1).

EXAMPLE 6.1. (Bohm-Aharonov effect) *The Example 5.1 of the Dirac monopole shows that the topology of the base space may require several local gauge potentials to describe a single global gauge field. In classical theory only the electromagnetic field was supposed to have physical significance while the potential was regarded as a mathematical artifact. In non-simply connected spaces the equation $dF = 0$ defines only a local potential but a global topological property of belonging to a given cohomology class. Bohm and Aharonov (*[AH1], [AH2], [WU3]*) suggested that in quantum theory the electromagnetic potential A_i in a multiply connected region of space-time has a further kind of significance that it does not have in the classical theory. They proposed to detect this topological effect by computing the phase shift $\oint A_i dx^i$ around a closed curve not homotopic to the identity and computing its effect on an electron interference experiment. The predicted effect was confirmed by experimental observations. This result firmly extablished the physical significance of the gauge potential.*

In this section we restrict ourselves to the space $\mathcal{A}$ of gauge connections. If ω is a gauge connection on P then in a local gauge $s \in \Gamma(U, P)$ we have the gauge potential

$$A_s = s^* \omega$$

and a gauge transformation g reduces to a G-valued function g_s on U, with action on (A_s, ϕ) given by

$$g_s \cdot A_s = (ad g_s) \circ A_s + g_s^* \Theta,$$

where Θ is the canonical 1-form on G and

$$g_s \cdot \phi = (ad g_s) \cdot \phi.$$

The gauge field F_ω is the unique 2-form on M satisfying

$$F_\omega = s_\Omega,$$

where Ω is the curvature of the gauge connection ω. Locally (i.e. on U)

$$F_\omega = d^\omega A_s = dA_s + \frac{1}{2}[A_s, A_s],$$

where the bracket is taken in the Lie algebra g .

It is always possible to introduce a Riemannian metric on vector bundles over M . M itself admits a Riemannian metric. We assume that metrics are chosen on M and the bundles over M and the norm is defined on sections of these bundles as an L^2-norm if M is not compact.

The *Yang-Mills action* $\mathcal{A}_{YM}$ is defined by

(6.1) $$\mathcal{A}_{YM}(\omega) = \frac{1}{8\pi^2} \int_M \|F_\omega\|^2 .$$

The corresponding pure *Yang-Mills equation* is

(6.2) $$\delta^\omega F_\omega = 0$$

which is equivalent to

(6.3) $$d^\omega * F_\omega = 0 .$$

Note that in this case there is only one nontrivial Bianchi identity

(6.4) $$d^\omega F_\omega = 0 .$$

This identity is a consequence of the Cartan structure equations and expresses the fact that locally, F is derived from a potential. It is customary to consider the pair (6.2) and (6.4) or (6.3) and (6.4) as the Yang-Mills equations. This is consistent with the fact that they reduce to the Maxwell equations for the electromagnetic field F when the gauge group G is $U(1)$ and M is the Minkowski space. When M is four dimensional we can associate to the pure Yang-Mills equations the first order *instanton* (resp.*anti-instanton*) *equations*

(6.5) $$F_\omega = *F_\omega \quad (\text{resp. } F_\omega = - * F_\omega).$$

The Bianchi identities imply that any solution of the instanton equations (6.5) is also a solution of the Yang-Mills equation (6.2). The fields satisfying $F_\omega = *F_\omega$ (resp. $F_\omega = - * F_\omega$) are called *self-dual* (resp. *anti-self-dual*) Yang-Mills fields. The solutions in the case of $M = S^4$ were called *instantons* , but this term is now used to denote any solution of the instanton equations (6.5) over a compact Riemannian manifold.

Almost (recently the existence of non-dual solutions of the Yang-Mills equations over S^4 has been extablished in [SI4]) all the solutions of the pure Yang-Mills equations that have been constructed to date are in fact solutions of the self-dual or anti-self- dual instanton equations. They are also called *pseudo-particle solutions.* The first such solutions, consisting of a 5 parameter family of self-dual Yang-Mills fields on $\mathbf{R}^4$, was constructed by Belavin, Polyakov, Schwartz, Tyupkin [BE1] in 1975 and is commonly referred to as the $BPST$ instanton. We will show that the $BPST$ instanton solutions correspond to the gauge group $SU(2)$ over the base manifold S^4 and have instanton number $k = 1$. In this case the *instanton number* k is defined by

$$k := -c_2(P(S^4, SU(2)))[S^4],$$

where c_2 denotes the second Chern class of the bundle P and $[S^4]$ denotes the fundamental cycle of the manifold S^4. The BPST solution was generalized to the so-called *multi-instanton solutions* which correspond to the self-dual Yang-Mills fields with instanton number k. A $5k$-parameter family of solutions was obtained by 't Hooft (unpublished) and a $(5k+4)$-parameter family was obtained by Jackiw, Nohl, Rebbi [JA2], [JA3], [JA4] in 1977. Other special solutions may be found in [BE2], [CH3]. For a given instanton number k the maximum number of parameters in the corresponding instanton solution can be i dentified with the dimension of the space of gauge inequivalent solutions (the moduli space). For $SU(2)$-instantons over S^4, this dimension of the moduli space was computed by Atiyah, Hitchin, Singer [AT8] by using the Atiyah-Singer index theorem and turns out to be $8k - 3$. Thus for $k = 1$ the moduli space has dimension 5 and the 5-parameter BPST solutions correspond to this space. For $k > 1$, the 't Hooft and Jackiw, Nohl, Rebbi solutions do not give all the possible instantons with instanton number k. An explicit construction of the full $(8k - 3)$-parameter family of solutions was given by Atiyah, Drinfeld, Hitchin and Manin [AT6]. An alternative construction was given by Atiyah and Ward [AT11] using the Penrose correspondence. Several solutions to the Yang-Mills equations on special manifolds of various dimensions have been obtained in [AC1], [BA1], [GR10], [LA1], [NO1], [SC3], [TR1]. Although their physical significance is not clear, they may prove to be useful in understanding the mathematical aspects of the Yang-Mills equations.

Particle motion in a Yang-Mills field

It is well known that Hamilton's equations of motion of a particle in classical mechanics can be given a geometrical formulation by using the phase space P of the particle. The phase space P is, at least locally, the cotangent space T^*Q of the configutration space Q of the particle. For a geometrical formulation of classical mechanics see R. Abraham, J. Marsden [bBA1]. It can be shown that this formalism can be extended to the motion of a charged particle in an electromagnetic field and leads to the usual equations with the Lorentz force. Now electromagnetic field is a gauge field with abelian gauge group $U(1)$. The orbits of contragradient action of $U(1)$ on the dual of its Lie algebra $u(1)$ are trivial. Identifying $u(1)$ and its dual with $\mathbf{R}$, we see that an orbit through $e \in \mathbb{R}$ is the point e itself. Thus in this case the choice of an orbit is the same thing as the choice of the unit of charge. This construction can be generalized to gauge fields with arbitrary structure group and in particular, to the Yang-Mills fields to obtain the equations of motion of a particle moving in a Yang-Mills field. A detailed discussion of this is given in V. Guillemin, S. Sternberg [bGU1]; further references are [MO3], [SH1], [ST2], [WE2].

6.2. Instantons and their moduli spaces

A complete set of solutions of instanton equations on S^4 was obtained by using the methods of algebraic geometry and the theory of complex manifolds. These methods have also been used in the study of the Yang-Mills equations over Riemann surfaces. Since this approach is not discussed here we simply indicate below some references where this and related topics are developed. They are [AT2], [AT3], [AT5], [AT11], [BO3], [BU1], [DO8], [DR1], [DR2], [FR3], [HA5], [MA4], [NA2].

Let M be a compact oriented Riemannian manifold of dimension 4. Let $P(M,G)$ be a principal bundle over M with compact semisimple Lie group G as structure group. A *G-instanton* (resp. *G-anti-instanton*) over M is a self-dual (resp. anti-self-dual) Yang-Mills field on the principal bundle P. The second Chern class and the Euler class of P are equal in this case and we define the *instanton number* k of a G-instanton by

$$k := -c_2(P)[M] = -\chi(P)[M].$$

Recall that if $F = F_\omega$ is the curvature form of a gauge connection ω, then we have

$$k := -c_2(P)[M] = -\frac{1}{8\pi^2}\int_M Tr(F \wedge F).$$

Decomposing F into its self-dual part F_+ and anti-self-dual part F_-, we get

$$k = \frac{1}{8\pi^2}\int_M (||F_+||^2 - ||F_-||^2).$$

Using F_+ and F_- we can rewrite (6.1) as follows:

$$\mathcal{A}_{YM}(\omega) = \frac{1}{8\pi^2}\int_M (||F_+||^2 + ||F_-||^2).$$

Comparing the above two equations we see that the Yang-Mills action is bounded below by the instanton number k, i.e.

$$\mathcal{A}_{YM}(\omega) \geq k, \ \forall \omega \in \mathcal{A}(P)$$

and the connections satisfying the instanton or the anti-instanton equations are the absolute minima of the action. Thus for a self-dual ($F = F_+$) Yang-Mills field, i.e. a self-dual G-instanton, we have

$$\mathcal{A}_{YM} = \frac{1}{8\pi^2}\int_M (||F||^2) = k.$$

In the rest of this section we restrict ourselves to considering self-dual instantons. The configuration space of these instantons is denoted by $C^+(P)$ and is defined by

$$C^+(P) := \{\omega \in \mathcal{A} \mid F_\omega = *F_\omega\},$$

where $\mathcal{A}$ is the space of gauge connections. The group $\mathcal{G}$ of gauge transformations acts on $C^+(P)$ and the quotient space under this action is called the *moduli space of k-instantons,* i.e. of self-dual instantons with instanton number k. It is denoted by $\mathcal{M}_k(M,G)$. Thus we have

$$\mathcal{M}_k(M,G) := C^+(P(M,G))/\mathcal{G}.$$

We now briefly indicate how the dimension of the moduli space $\mathcal{M}_k(M,G)$ can be computed by applying the Atiyah-Singer index theorem. The fundamental elliptic complex comes from a modification of the generalized de Rham sequence. For a vector bundle E over M associated to the principal bundle $P(M,G)$, the generalized de Rham sequence can be written as follows:

$$0 \to \Lambda^0(M,E) \xrightarrow{d^\omega} \Lambda^1(M,E) \xrightarrow{d^\omega} \Lambda^2(M,E) \xrightarrow{d^\omega} \cdots,$$

where $\Lambda^p(M,E) = \Gamma(\Lambda^p(T^*M) \otimes E)$. On the oriented Riemannian 4-manifold M the space $\Lambda^p(T^*M)$ splits under the action of the Hodge $*$ operator into a direct sum of self-dual and anti-self-dual 2-forms. This splitting extends to $\Lambda^2(M,E)$ so that

$$\Lambda^2(M,E) = \Lambda^2_+(M,E) \oplus \Lambda^2_-(M,E),$$

where $\Lambda^2_+(M,E)$ (resp. $\Lambda^2_-(M,E)$) is the space of self-dual (resp. anti-self-dual) 2-forms with values in the vector bundle E. Let

$$p_\pm : \Lambda^2(M,E) \to \Lambda^2_\pm(M,E)$$

be the canonical projections and define

$$d^\omega_- : \Lambda^1(M,E) \to \Lambda^2_-(M,E) \text{ by } d^\omega_- := p_- \circ d^\omega.$$

Then we have

$$
0 \to \Lambda^0(M,E) \xrightarrow{d^\omega} \Lambda^1(M,E) \xrightarrow{d^\omega} \Lambda^2(M,E) \xrightarrow{d^\omega} \cdots, \quad \Lambda^2(M,E) \xrightarrow{p_-} \Lambda^2_-(M,E) \tag{6.6}
$$

For a self-dual gauge connection ω,

$$
d^\omega_- \circ d^\omega = \Omega^\omega_- = 0. \tag{6.7}
$$

Using equation (6.7) and the diagram (6.6) we obtain the fundamental elliptic complex

$$
0 \to \Lambda^0(M,E) \xrightarrow{d^\omega} \Lambda^1(M,E) \xrightarrow{d^\omega_-} \Lambda^2_-(M,E) \to 0.
$$

For $E = ad(P) = \mathcal{H} = P \times_{ad} \mathfrak{g}$, we define the *twisted Dirac operator*

$$
\mathcal{D} : \Lambda^1(M,E) \to \Lambda^0(M,E) \oplus \Lambda^2_-(M,E) \quad \text{by} \quad \alpha \mapsto \delta^\omega \alpha \oplus d^\omega_- \alpha,
$$

where δ^ω is the formal adjoint of d^ω. It can be shown that the twisted Dirac operator $\mathcal{D}$ is elliptic and that its index $Ind(\mathcal{D})$ equals the dimension of the moduli space $\mathcal{M}_k(M,G)$. This index can be computed by using the Atiyah-Singer index theorem and leads to the following result

$$
\begin{aligned}
\dim \mathcal{M}_k(M,G) &= Ind(\mathcal{D}) = \\
&= 2\,ch(P \times_{ad} \mathfrak{g}_{\mathbb{C}})[M] - \\
&- \frac{1}{2} \dim G(\chi(M) - sign(M)),
\end{aligned} \tag{6.8}
$$

where ch is the Chern character, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is the complexification of the Lie algebra $\mathfrak{g}$, $\chi(M)$ is the Euler characteristic of M and $sign(M)$ is the Hirzebruch signature of M. If $M = S^4$ then $\chi(M) = 2$ and $sign(M) = 0$, thus the formula (6.8) reduces to

$$
\dim \mathcal{M}_k(S^4,G) = 2\,ch(P \times_{ad} \mathfrak{g}_{\mathbb{C}})[S^4] - \dim G. \tag{6.9}
$$

We apply this formula to some standard non-abelian gauge groups.

(1) $G = SU(n)$, $n \geq 2$. Then $\dim G = n^2 - 1$ and $ch(P \times_{ad} \mathfrak{g}_{\mathbb{C}})[S^4] = 2nk$, where $k = -c_2(P)$ is the instanton number. Thus (6.9) becomes

$$
\dim \mathcal{M}_k(S^4, SU(n)) = 4nk - n^2 + 1 \tag{6.10}
$$

Applying the formula (6.10) to the particular case of $G = SU(2)$ we obtain

$$\dim \mathcal{M}_k(S^4, SU(2)) = 8k - 3.$$

For instanton number $k = 1$, we get

$$\dim \mathcal{M}_1(S^4, SU(2)) = 5,$$

corresponding to the BPST family of solutions. In the next section we give an explicit geometric construction of the BPST instantons and indicate briefly the construction of the ($8k - 3$)-parameter family of instantons with instanton number k.

(2) $G = Spin(n), n > 3$, where $Spin(n)$ is the universal covering group of $SO(n)$. Then $\dim G = n(n-1)/2$ and $ch(P \times_{ad} \mathfrak{g}_{\mathbb{C}})[S^4] = 2(n-2)k$, where $k = -c_2(P)$ is the instanton number. Thus (6.9) becomes

$$\dim \mathcal{M}_k(S^4, Spin(n)) = 4(n-2)k - n(n-1)/2, \; k \geq n/2. \tag{6.11}$$

For small values of n some of the gauge groups are locally isomorphic. For example $Spin(6)$ is locally isomorphic to $SU(4)$. Thus (6.10) and (6.11) lead to the same dimension for the moduli space in this case.

(3) $G = Sp(n)$. Then $\dim G = n(2n+1)$ and $ch(P \times_{ad} \mathfrak{g}_{\mathbb{C}})[S^4] = 2(n+1)k$, where $k = -c_2(P)$ is the instanton number. Thus

$$\dim \mathcal{M}_k(S^4, Sp(n)) = 4(n+1)k - n(2n+1), \; k \geq n/4.$$

6.3. BPST instantons

It can be shown that the BPST instanton solution over the base manifold $\mathbb{R}^4$ can be extended to the conformal compactification of $\mathbb{R}^4$, i.e. S^4. This extension is characterized by a self-dual connection in a non-trivial $SU(2)$-bundle over S^4. This bundle is the quaternionic Hopf fibering of $\mathbb{H}^2$ over $\mathbb{HP}^1$, where $\mathbb{H}$ is the space of quaternions and $\mathbb{HP}^1$ is the quaternionic projective space of lines through the origin in the quaternionic plane $\mathbb{H}^2$. We identify $\mathbb{H}$ with $\mathbb{R}^4$ by the map $\mathbb{H} \to \mathbb{R}^4$ given by

$$x = x_0 + ix_1 + jx_2 + kx_3 \longmapsto (x_0, x_1, x_2, x_3).$$

Then $\mathbb{H}^2$ is isomorphic to $\mathbb{R}^8$ and each quaternionic line intersects the 7-sphere $S^7 \subset \mathbb{H}^2$ in S^3. On the other hand the base $\mathbb{HP}^1$ can be identified with S^4. Thus the quaternionic Hopf fibering leads to the bundle

$$\begin{array}{c} S^7 \\ \downarrow S^3 \\ S^4 \end{array}$$

The fiber S^3 can be identified with the group $SU(2)$ of unit quaternions and its action on $\mathbf{H}^2$ by multiplication restricts to S^7 making it a principal $SU(2)$-bundle over S^4. This follows from the observation that $\alpha \in SU(2)$ and $(x, y) \in S^7 \subset \mathbf{H}^2$ imply that $(\alpha x, \alpha y) \in S^7$. This principal bundle is clearly non-trivial and admits a canonical connection ω_1 (also called the universal connection) whose curvature Ω_1 is self-dual and hence satisfies the Yang-Mills equations. It corresponds to a BPST instanton of instanton number 1. The entire BPST family of instantons can be generated from this solution as follows. The group $SO(5,1)$ acts on S^4 by conformal transformations and this action induces an action on ω_1. If $g \in SO(5,1)$, then we denote the induced action on ω_1 also by g. Then $g\omega_1$ is also an $SU(2)$-connection over S^4 and it has self-dual curvature. Since the Yang-Mills action is conformally invariant, the solution generated by $g\omega_1$ also has instanton number 1. The connection $g\omega_1$ is gauge equivalent to ω_1 (and hence determines the same point in the moduli space) if and only if g is an isometry of S^4, i.e. if and only if $g \in SO(5) \subset SO(5,1)$. Thus the space of gauge inequivalent, self-dual, $k = 1$, $SU(2)$-connections on S^4, or the moduli space $\mathcal{M}_1(S^4, SU(2))$, is given by

$$\mathcal{M}_1(S^4, SU(2)) = SO(5,1)/(SO(5)).$$

We note that the quotient space $SO(5,1)/(SO(5))$ can be identified with the hyperbolic 5-space H_5. We now give an explicit local formulation of the BPST family. Consider the chart defined by the map ψ_{e_5}

$$\psi_{e_5} : S^4 - \{e_5\} \to \mathbf{R}^4,$$

where ψ_{e_5} is obtained by projecting from $e_5 = (0,0,0,0,1) \in \mathbf{R}^5$ onto the tangent hyperplane to S^4 at $-e_5$. This chart gives conformal coordinates on S^4. Identifying $\mathbf{R}^4$ with $\mathbb{H}$, the metric can be written as

$$ds^2 = 4\,|dx|^2/(1 + |x|^2),$$

where $|x|^2 = x\bar{x}$, $|dx|^2 = dx d\bar{x}$. Identifying the Lie algebra $su(2)$ as the set of pure imaginary quaternions, we can write down the gauge potential $A^{(1)}$, corresponding to the universal connection ω_1, as follows:

$$A^{(1)} = Im\left(\frac{x d\bar{x}}{1 + |x|^2}\right)$$

It is possible to give a similar expression for the potential in the chart obtained by projecting from $-e_5$ and to show that these expressions are compatible under change of chart and hence define a global connection with corresponding Yang-Mills field. However, we apply the removable singularities theorem of Uhlenbeck [UH1] to guarantee the extension of the local connection to all of S^4. This procedure is also useful in the general construction of multi-instantons where there is a finite number of removable singularities. The Yang-Mills field $F^{(1)}$ corresponding to the gauge potential $A^{(1)}$ is given by

$$F^{(1)} = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}.$$

Using the formula

$$vol(S^{2m}) = \frac{2^{2m+1}\pi^m m!}{(2m)!}$$

and calculating the Euclidean norm $||F^{(1)}||^2_{Eu} = 3$ we can evaluate the Yang-Mills action

$$\mathcal{A}_{YM} = \frac{1}{8\pi^2}\int_{S^4} ||F^{(1)}||^2_{Eu} = 3 \cdot vol(S^4)/(8\pi^2) = 1.$$

On the other hand $\mathcal{A}_{YM} = k$ for a self-dual instanton with instanton number k. Thus we see that the above solution corresponds to $k = 1$. We now apply the conformal dilatation

$$f_\lambda : \mathbf{R}^4 \to \mathbf{R}^4 \text{ defined by } x \mapsto x/\lambda, \qquad 0 < \lambda < 1$$

to obtain induced connections $A^{(\lambda)} = f_\lambda^* A^{(1)}$ and the corresponding Yang-Mills fields $F^{(\lambda)} = f_\lambda^* F^{(1)}$. Because of conformal invariance of the action $\mathcal{A}_{YM}$, the fields $F^{(\lambda)}$ have the same instanton number $k = 1$. The local expressions for $A^{(\lambda)}$ and $F^{(\lambda)}$ are given by

$$A^{(\lambda)} = Im\left(\frac{x d\bar{x}}{\lambda^2 + |x|^2}\right),$$

and

$$F^{(\lambda)} = \frac{\lambda^2 dx \wedge d\bar{x}}{(\lambda^2 + |x|^2)^2}.$$

If we write the above expressions in terms of the quaternionic components, we recover the formulas for the BPST instanton. The connections corresponding to different

λ are gauge-inequivalent, since $||F||^2$ is a gauge invariant function. Moreover choosing an arbitrary point $q \in S^4$ and considering the chart obtained projecting from this point, we obtain gauge inequivalent connections for different choices of q. In fact we have a 5-parameter family of instantons parametrized by (q, λ) where $q \in S^4$ and $\lambda \in (0,1)$ and (q_1, λ_1) and (q_2, λ_2) give gauge-equivalent connections iff $q_1 = q_2$ and $\lambda_1 = \lambda_2$. It is customary to call q the *center* of the instanton and λ its *size*. The map

$$(q, \lambda) \mapsto (1 - \lambda) q$$

from $S^4 \times (0,1)$ into $\mathbb{R}^5$ is an isomorphism onto the punctured open ball $B^5 - \{0\}$. The universal connection $A^{(1)}$ corresponds to the origin. Thus the moduli space of gauge-inequivalent instantons is identified with the open unit ball B^5, which is the Poincaré model of the hyperbolic 5-space H_5. The connection corresponding to (q, λ) as $\lambda \to 0$ can be identified with a boundary point of the open ball B^5. This realizes S^4 as the boundary of the ball B^5. Thus our base space appears as the boundary of the moduli space. This is one of the key ideas of Donaldson in his work on the topology of the moduli space of instantons [DO3]. The moduli space of the fundamental BPST instantons or self-dual $SU(2)$ Yang-Mills fields with instanton number 1 over the Euclidean 4-sphere S^4, is denoted by $\mathcal{M}_1^+$. It can be shown [AT8] that the action of the group $SO(5,1)$ of conformal diffeomorphisms of S^4 induces a transitive action of $SO(5,1)$ on the moduli space $\mathcal{M}_1^+$ with isotropy group $SO(5)$. Thus $\mathcal{M}_1^+$ is diffeomorphic to the homogeneous hyperbolic 5-space $SO(5,1)/SO(5)$. In particular, the topology of $\mathcal{M}_1^+$ is the same as that of $\mathbb{R}^5$. A more general result of Donaldson (see e.g. [DO3], D.S. Freed, K.K. Uhlenbeck [bFR1], H.B. Lawson, Jr. [bLA2]) shows that any 1-connected 4-manifold M with positive-definite intersection form, can be realized as a boundary of a suitable moduli space. The instanton solutions with instanton number 1 can be used to construct instantons with instanton number k. These solutions are sometimes referred to as k-instantons or multi-instantons. The first such solutions were obtained by physicists using a non-linear analog of the superposition principle applied to k suitably placed 1-instantons. A $5k$-parameter family of solutions of instanton equations with instanton number k was obtained by 't Hooft by a method which may be described roughly as the supersposition of k non-interacting instantons of instanton number 1. Thus the parameter λ is replaced by a vector $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ and $q \in \mathbb{R}^4 = \mathbb{H}$ is replaced by a vector $q = (q_1, \ldots, q_k) \in \mathbb{H}^k$. Jackiw, Nohl, Rebbi obtained a $(5k + 4)$-parameter family, $k > 2$, by a slightly different method. Instead of describing these solutions we will describe briefly the construction of the most general $(8k - 3)$-parameter family of solutions due to Atiyah [bAT1]. It turns out that the general solution is obtained by starting with a vector $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{H}^k$ and a $k \times k$ symmetric matrix B over $\mathbb{H}$ satisfying the following conditions

(i) $(B^\dagger B + \lambda^\dagger \lambda)$ is a real matrix ($\dagger$ is the quaternionic conjugate transpose),

(ii) $rank \begin{pmatrix} \lambda \\ B - xI \end{pmatrix} = k, \qquad \forall x \in \mathbb{H}$.

The local expression for the gauge potential $A^{(\lambda,B)}$ determined by the pair (λ, B) , $\lambda \in \mathbb{H}^k$, $B \in \mathbb{H}_{k\times k}$ is given by

$$A^{(\lambda,B)} = f^*(A(u)), \text{ where } u = (u_1, \dots, u_k) \in \mathbb{H}^k,$$

and $f : \mathbb{H} \to \mathbb{H}^k$ is defined by

$$f(x) = [\lambda(B - xI)^{-1}]^\dagger, \quad \forall x \in \mathbb{H}$$

and $A(u)$ is the $SU(2)$-gauge potential on the space $\mathbb{H}^k$ given by

$$A(u) = Im\left(\frac{udu^\dagger}{1 + |u|^2}\right).$$

The gauge field $F^{(\lambda,B)}$ corresponding to the gauge potential $A^{(\lambda,B)}$ contains a finite number of removable singularities and hence, by applying Uhlenbeck's theorem on removable singularities, the solution can be extended smoothly to S^4 . The potentials determined by (λ, B) and (λ', B') are gauge equivalent iff there exists $\alpha \in SU(2)$ and $T \in O(k)$ such that

$$\lambda' = \alpha\lambda T \qquad \text{and} \qquad B' = T^{-1}BT.$$

We may carry out a naive counting of free real parameters as follows. The pair (λ, B) gives

$$4k + 4(\frac{1}{2}k(k+1)) = 2k^2 + 6k \tag{6.12}$$

real parameters. The reality condition (i) above, involves $3k(k-1)/2$ parameters. The condition (ii) on the rank does not restrict the number of parameters. The gauge equivalence further reduces the number of parameters in (6.12) by 3 (by the $SU(2)$-action) and by $k(k-1)/2$ (by the $O(k)$-action). Thus we have the following count for free real parameters.

$$2k^2 + 6k - 3k(k-1)/2 - 3 - k(k-1)/2 = 8k - 3.$$

It can be shown that this family of $(8k-3)$-parameter solutions exhausts all the possible solutions up to gauge equivalence. Thus the space of these solutions may be identified as the moduli space $\mathcal{M}_k(S^4, SU(2))$ of k-instantons (i.e. instantons with instanton number k) over S^4 . In the theorem below we give an alternative characterization of this moduli space.

THEOREM 6.2. *Let* k *be a positive integer. Write*

$$B(q) = A_1 q_1 + A_2 q_2,$$

where,

$$q = (q_1, q_2) \in \mathbb{H}^2, \text{ and } A_i \in Hom_{\mathbb{H}}(\mathbb{H}^k, \mathbb{H}^{k+1}), i = 1, 2.$$

Define the manifold M *and Lie group* G *by*

$$M = \{(A_1, A_2) \mid B(q)^\dagger B(q) \in GL(k, \mathbf{R}), \forall q \neq 0\},$$

$$G = (Sp(k+1) \times GL(k, \mathbf{R}))/\mathbf{Z}_2.$$

Then $M \neq \emptyset$, G *acts freely and properly on* M *by the action*

$$(a, b) \cdot (A_1, A_2) := (aA_1 b^{-1}, aA_2 b^{-1}),$$

and the quotient space M/G *is a manifold of dimension* $8k - 3$ *which is isomorphic to the moduli space* $\mathcal{M}_k(S^4, SU(2))$ *of instantons of instanton number* k.

6.4. Geometry of the Yang-Mills moduli space

Study of the differential geometric and topological aspects of the moduli space of Yang-Mills instantons on a four dimensional manifold was initiated by Donaldson [DO4]. Building on the analytical work of Taubes [TA4], [TA5], [TA6], and Uhlenbeck [UH1], [UH2], [UH3] Donaldson studied the space $\mathcal{M}_1^+(M)$, where M is a compact, simply connected, differential 4-manifold with positive definite intersection form. He showed that for such M the intersection form is equivalent to the unit matrix. Freedman had proved a classification theorem for topological 4-manifolds [FR1] which shows that every positive definite form occurs as the intersection form of a topological manifold. A spectacular application of this classification theorem is his proof of the 4-dimensional Poincaré conjecture. Donaldson's result showed the profound difference in the differentiable and topological cases in dimension 4 in striking contrast to the known results in dimensions greater than 4. In particular, these results imply the existence of exotic 4-spaces which are homeomorphic but not diffeomorphic to the standard Euclidean 4-space R^4. Soon many examples of such exotic $\mathbf{R}^4$'s were found [GO3], [GO4]. A recent result of Taubes gives an uncountable family of exotic $\mathbf{R}^4$'s and yet this list of examples is not exhaustive. Using instantons as a powerful new tool Donaldson has opened up a new area of what may be called gauge theoretic mathematics (see [BO8], [BO9], [DO2]-[DO11], [FI1], [FL1], [NA1], [ST1], [TA6]).

In the Feynman path integral approach to quantum field theory, one is interested in integrating a suitable function of the classical action over the space of all gauge inequivalent fields. In addition, one assumes that one can make an «analytic continuation» from the Lorentz manifold to a Riemannian manifold, carry out the integration and then transfer the results back to the physically relevant space-time manifold. Although the mathematical aspects of this program are far from clear, it has served as a motivation for the study of Euclidean Yang-Mills fields i.e. fields over a Riemannian base manifold. Thus for the quantization of Yang-Mills field, the space over which the Feynman integral is to be evaluated turns out to be the Yang-Mills moduli space. Evaluation of such integrals requires a detailed knowledge of the geometry of the moduli space. We have very little information on the geometry of the general Yang-Mills moduli space. However, we know that the dominant contribution to the Feynman integral comes from solutions which absolutely minimize the Yang-Mills action, i.e from the instanton solutions. If $\mathcal{Y}$ denotes the Yang-Mills moduli space, then $\mathcal{Y} = \cup \mathcal{Y}_k$, where $\mathcal{Y}_k$ is the moduli space of fields with instanton number (the first Pontryagin class) k. The moduli space $\mathcal{M}_k^+$ of self-dual Yang-Mills fields or instantons of instanton number k is a subspace of $\mathcal{Y}_k$. Thus one hopes to obtain some information by integrating over the space $\mathcal{M}_k^+$. Recently several authors ([GR8], [GR9], [DO1]) have undertaken the study of the geometry of the space $\mathcal{M}_k^+$ and we now have detailed results about the Riemannian metric, volume form and curvature of the most basic moduli space $\mathcal{M}_1^+$. We give below a brief discussion of these results.

Let (M, g) be a compact, oriented, Riemannian 4-manifold. Let $P(M, G)$ be a principal G-bundle, where G has a bi-invariant metric h. The metrics g and h induce the inner products $\langle\ ,\ \rangle_{(g,h)}$ on the spaces $\Lambda^k(M, ad\, P)$ of k-forms with values in the vector bundle $ad\, P$. We can use these inner products to define a Riemannian metric on the space of gauge connections $\mathcal{A}(P)$ as follows. Recall that the space $\mathcal{A}(P)$ is an affine space, so that for each $\omega \in \mathcal{A}(P)$ we have the canonical identification between the tangent space $T_\omega \mathcal{A}(P)$ and $\Lambda^1(M, ad\, P)$. Using this identification we can transfer the inner product $\langle\ ,\ \rangle_{(g,h)}$ on $\Lambda^1(M, ad\, P)$ to $T_\omega \mathcal{A}(P)$. Integrating this pointwise inner product against the Riemannian volume form we obtain an L^2 inner product on the space $\mathcal{A}(P)$. This inner product is invariant under the action of the group $\mathcal{G}$ on $\mathcal{A}$ and hence we get an inner product on the moduli space $\mathcal{A}/\mathcal{G}$. Recall that $\mathcal{G}$ does not act freely on $\mathcal{A}$, but $\mathcal{G}/Z(\mathcal{G})$ acts freely on the open dense subset $\mathcal{A}_{ir}$ of irreducible connections. After suitable Sobolev completions of all the relevent spaces, the moduli space $\mathcal{O}_{ir} = \mathcal{A}_{ir}/\mathcal{G}$ of irreducible connections can be given the structure of a Hilbert manifold. The L^2 metric on $\mathcal{A}$ restricted to $\mathcal{A}_{ir}$ induces a weak Riemannian metric on $\mathcal{O}_{ir}$ by requiring the canonical projection $\mathcal{A}_{ir} \to \mathcal{O}_{ir}$ to be a Riemannian submersion. The space $\mathcal{I}_k$ of irreducible instantons of instanton number $k = -c_2(P)$, is defined by $\mathcal{I}_k = \mathcal{M}_k^+ \cap \mathcal{O}_{ir}$, where $\mathcal{M}_k^+$ is the moduli space of all instantons of instanton number k. The space $\mathcal{I}_k$ is a finite dimensional manifold with singularities and the weak Riemannian metric on $\mathcal{O}_{ir}$ restricts to a Riemannian metric on $\mathcal{I}_k$.

The space $\mathcal{I}_1$ was studied by Donaldson [DO3] in the case when M is a simply connected manifold with positive definite intersection form and when the gauge group $G = SU(2)$. He proved that there exists a compact set $K \in \mathcal{I}_1$ such that $\mathcal{I}_1 - K$ is a union of a finite number of components, one of which is diffeomorphic to $M \times (0,1)$ and all the others are diffeomorphic to $\mathbb{C}\mathbf{P}^2 \times (0,1)$. Using the Riemannian geometry of $\mathcal{I}_1$ discussed above we can put this result in its geometric perspective as follows.

THEOREM 6.3. *Let M be a simply connected manifold with positive definite intersection form and let the gauge group $G = SU(2)$. Then*

i) $\mathcal{I}_1$ is an incomplete manifold with finite diameter and volume.

ii) Let $\bar{\mathcal{I}}_1$ be the metric completion of $\mathcal{I}_1$. Then $\bar{\mathcal{I}}_1 - \mathcal{I}_1$ is the disjoint union of a finite set of po ints $\{p_i\}$ and a set X diffeomorphic to M. There exists an $\epsilon > 0$ such that $M \times [0,\epsilon)$ is diffeomorphic to a neighborhood of X in $\bar{\mathcal{I}}_1$ with the pullback metric asymptotic to a product metric and for each p_i there is a neighborhood diffeomorphic to $\mathbb{C}\mathbf{P}^2 \times (0,\epsilon)$ with the pullback metric asymptotic to a cone metric with base $\mathbb{C}\mathbf{P}^2$. ∎

See [GR8], [GR9] for a proof and further details.

In general, the L^2 metric on a moduli space cannot be calculated explicitly, since it depends on global analytic data about M. However, for the fundamental BPST instanton moduli space $\mathcal{I}_1$, the metric and the curvature have recently been computed by several people (see, for example, [DO1], [GR8], [HA1], [IT3]). We have the following theorem.

THEOREM 6.4. *There exists a diffeomorphism $\phi : \mathbf{R}^5 \to \mathcal{I}_1$ for which the pullback metric has the form*

$$\phi^* g_1 = \psi^2(r) g,$$

where g_1 is the L^2 metric on $\mathcal{I}_1$ and g is the standard Euclidean metric on $\mathbf{R}^5$, and ψ is a smooth function of the distance r. ∎

The explicit formula for the function ψ is quite complicated and is given in [GR8]. The moduli space of Yang-Mills connections on various base manifolds has recently been studied in [IT1], [IT2], [IT3], [GR7], [GR8], [GR9]. We now state the result on the curvature of the moduli space $\mathcal{M}_{ir}$ of irreducible instantons on an arbitrary Riemannian manifold with compact gauge group. The calculation of this curvature is based on a generalization of O'Neill's formula [ON1] for the curvature of a Riemannian submersion.

THEOREM 6.5. *Let $X, Y \in T_\alpha \mathcal{A}$ be the horizantal lifts of tangent vectors $X_1, Y_1 \in T_{[\alpha]}\mathcal{M}_{ir}$, where $[\alpha]$ is the equivalence class of gauge connections that are gauge equivalent to α. Then the sectional curvature R of $\mathcal{M}_{ir}$ at $[\alpha]$ is given by*

$$\begin{aligned}\langle R(X_1,Y_1)Y_1,X_1 \rangle = 3\langle b_X^*(Y), G_\alpha^0(b_X^*(Y)) \rangle + \\ + \langle b_X^-(X), G_\alpha^2(b_Y^-(Y)) \rangle - \\ - \langle b_X^-(Y), G_\alpha^2(b_X^-(Y)) \rangle\end{aligned}$$

where b is bracketing on bundle valued forms, b^* its adjoint and b^- is b followed by orthogonal projection onto $\Lambda_-^2(M, ad\, P)$ and G^i are the Green operators of the corresponding Laplacians on bundle valued forms. ∎

7. YANG-MILLS-HIGGS FIELDS

7.1. Yang-Mills-Higgs fields

Let (M,g) be a compact Riemannian manifold and $P(M,G)$ a principal bundle with compact semi-simple gauge group G. Let h denote a fixed bi-invariant metric on G. The metrics g and h induce inner products and norms on various bundles associated to P and their sections. We denote all these different norms by the same symbol, since the particular norm used is clear from the context. The *Yang-Mills-Higgs configuration space* is defined by

$$C := \mathcal{A}(P) \times \Gamma(ad\, P).$$

i.e.

$$C := \{(\omega, \phi) \in \Lambda^1(P, \mathbf{g}) \times \Lambda^0(M, ad\, P)\}$$

where ω is a gauge connection and ϕ is a section of the *Higgs bundle* $ad\, P$.

A *Yang-Mills-Higgs action with self-interaction potential* $V : \mathbf{R}^+ \to \mathbf{R}^+$, is defined on the configuration space C by

$$\mathcal{A}_S(\omega, \phi) = c(M) \int_M [\, ||F_\omega||^2 + c_1 ||d^\omega \phi||^2 + c_2 V(||\phi||^2)] \tag{7.1}$$

where $c(M)$ is a *normalizing constant* that depends on the dimension of M ($c(M) = \frac{1}{8\pi^2}$ for a four dimensional manifold) and c_1, c_2 are the *coupling constants.* The constant c_1 measures the relative strengths of the gauge field and the gauge field-Higgs field interaction. When $c_1 \neq 0$, the constant c_2/c_1 measures the relative strengths of the

Higgs field self-interaction and the gauge field-Higgs field interaction. A *Yang-Mills-Higgs system with self-interaction potential* V is defined as a critical point $(\omega, \phi) \in C$ of the action $\mathcal{A}_S$. The corresponding Euler-Lagrange equations are

(7.2a) $$\delta^\omega F_\omega + c_1[\phi, d^\omega \phi] = 0,$$

(7.2b) $$\delta^\omega d^\omega \phi + c_2 V'(||\phi||^2)\phi = 0,$$

where $V'(x) = dV/dx$. Equations (7.2) are called the *Yang-Mills-Higgs field equations with self-interaction potential* V. In the physics literature it is customary to define the current J by

(7.3a) $$J = -c_1[\phi, d^\omega \phi].$$

Using this definition of the current we can rewrite equation (7.1a) as follows:

(7.3b) $$\delta^\omega F_\omega = J.$$

Note that in these equations δ^ω is the formal L^2-adjoint of the corresponding map d^ω. Thus in (7.2a) δ^ω is a map

$$\delta^\omega : \Lambda^2(M, ad\, P) \rightarrow \Lambda^1(M, ad\, P)$$

and in (7.2b) δ^ω is a map

$$\delta^\omega : \Lambda^1(M, ad\, P) \rightarrow \Lambda^0(M, ad\, P) = \Gamma(ad\, P).$$

The pair (ω, ϕ) also satisfies the following *Bianchi identities*

(7.4a) $$d^\omega F_\omega = 0,$$

(7.4b) $$d^\omega \cdot d^\omega \phi = [F_\omega, \phi].$$

Note that these identities are always satisfied whether or not equations (7.1) are satisfied.

We note that no solution in closed form of the Yang-Mills-Higgs equations with self-interaction potential is known for $c_2 > 0$, but existence of spherically symmetric solutions is known. Also existence of solutions for the system is known in dimensions 2 and 3. It can be shown that the solutions of equations (7.2) on R^n satisfy the following relation A. Jaffe, C. Taubes [bJA1].

$$(n-4)||F_\omega||^2 + (n-2)c_1||d^\omega \phi||^2 + nc_2||V(||\phi||^2)|| = 0.$$

From this relation it follows that for $c_1 \geq 0$, $c_2 \geq 0$ there are no nontrivial solutions for $n > 4$ and for $n = 4$ every solution decouples (i.e. is equivalent to a pure Yang-Mills solution.) For the Yang-Mills-Higgs system on R^3 with $c_1 = 1$ and $c_2 \geq 0$ we have the following result [GR5].

THEOREM 7.1. *Let* $V(t) = (1-t)^2(1+at)$, $a \geq 0$. *Then for* c_2 *sufficiently small, there exists a positive action solution to equations (7.2) which is not gauge equivalent to a spherically symmetric solution. Furthermore, for* c_2 *smaller still, there exists a solution which has the above properties and is, in addition, not a local minimum of the action.* ∎

Writing $c_1 = 1$ and $c_2 = 0$ in (6.13) we get the usual Yang-Mills-Higgs action

$$\mathcal{A}_{YMH}(\omega,\phi) = c(M)\int_M [\,||F_\omega||^2 + ||d^\omega\phi||^2\,]. \tag{7.5}$$

The corresponding *Yang-Mills-Higgs equations* are

$$\delta^\omega F_\omega + [\phi, d^\omega\phi] = 0, \tag{7.6a}$$

$$\delta^\omega d^\omega \phi = 0. \tag{7.6b}$$

7.2. Dimensional reduction

At least locally the Yang-Mills-Higgs equations (7.6) may be thought of as obtained by dimensional reduction from the pure Yang-Mills equations. To see this consider a Yang-Mills connection α on the trivial principal bundle $\mathbf{R}^{m+1} \times G$ over $\mathbf{R}^{m+1}$ and let

$$A_i dx^i + A_{m+1} dx^{m+1}, \quad 1 \leq i \leq m$$

be the gauge potential on $\mathbf{R}^{m+1}$ with values in $\mathbf{g}$. Suppose that this potential does not depend on x^{m+1}. Define $\phi := A_{m+1}$, then ϕ can be regarded as a Higgs potential on $\mathbf{R}^m$ with values in $\mathbf{g}$ and

$$A = A_i dx^i, \quad 1 \leq i \leq m$$

can be regarded as the gauge potential on $\mathbf{R}^m$ with values in $\mathbf{g}$. Let ω denote the gauge connection on $\mathbf{R}^m \times G$ corresponding to A. Then we have

$$(F_\omega)_{ij} = (F_\alpha)_{ij}, \quad (F_\alpha)_{i\ m+1} = (d^\omega\phi)_i \ 1 \leq i,j \leq m$$

and the pure Yang-Mills action of F_α on $\mathbf{R}^{m+1}$ reduces to the Yang-Mills-Higgs action (7.5). It is easy to see that this reduction is a consequence of the translation invariance of the pure Yang-Mills system in the x^{m+1}-direction. In general, suppose that T is a Lie group which acts on $P(M,G)$ as a subgroup of $Diff_M(P)$ with induced action on M. We say that T is a *symmetry group of the connection* ω on P or that ω is a T*-invariant connection* on P if the following condition is satisfied.

$$L_{\hat{A}}\omega = 0, \forall A \in \mathbf{t},$$

where $\hat{A}$ is the fundamental vector field corresponding to the element A in the Lie algebra $\mathbf{t}$ of T. Under certain conditions P/T is a principal bundle over M/T and a system of equations on the original bundle can be reduced to a coupled system on the reduced base. Reduction of such systems is discussed in [FO3], [HA2], [HA3]. For a particular class of self interaction potentials such reduction and the consequent symmetry breaking are responsible for the Higgs mechanism [HI2]. For a geometric formulation symmetry breaking and Higgs mechanism see [FU1], [KE1]. For a geometrical description of dimensional reduction and its relation to the Kaluza-Klein theories, see [CO1], [JA5] and the forthcoming book by R. Coquereaux and A. Jadczyk [bCO1]. For a relation with symplectic reduction see [SH1].

7.3. Monopoles

Finite action solutions of equations (7.1) are called *solitons*. Locally these solutions correspond to time-independent finite energy solutions on $\mathbf{R} \times M$. The Yang-Mills instantons discussed earlier are soliton solutions of equations (7.1) corresponding to $\phi = 0$, $c_2 = 0$. The soliton solutions on a 2-dimensional base are known as *vortices* and those on a 3-dimensional base are known as *monopoles*. Vortices and monopoles have many properties that are qualitatively similar to those of the Yang-Mills instantons.

When M is three dimensional, we can Associate to the Yang-Mills-Higgs equations the first order *Bogomolnyi equations* [BO1]

$$F_\omega = \pm * d^\omega \phi. \tag{7.7}$$

Equation (7.7) is also referred to as the *monopole equation*. The Bianchi identities (6.15) imply that each solution (ω, ϕ) of the Bogomolnyi equations (6.18) is a solution of the second order Yang-Mills-Higgs equations. In fact such solutions of the Yang-Mills-Higgs equations are global minima on each connected component of the Yang-Mills-Higgs *monopole configuration space* C_m defined by

$$C_m = \{(\omega, \phi) \in C \mid \mathcal{A}_S(\omega, \phi) < \infty, \lim_{y\to\infty} \sup_{|x|\geq y} |1 - \|\phi\|| = 0\}.$$

Locally the Bogomolnyi equations are obtained by applying the reduction procedure of section 7.2 to the instanton or anti-instanton equations on $\mathbf{R}^4$. No such first order equations corresponding to the Yang-Mills-Higgs equations with self-interaction potential are known. In particular, a class of solutions of the Bogomolnyi equations on R^3

has been studied extensively. They are the most extensively studied special class of the Yang-Mills monopole solutions. If the gauge group is a compact simple Lie group, then every solution (ω, ϕ) of the Bogomolnyi equations satisfying certain asymptotic conditions defines a gauge invariant set of integers. These integers are topological invariants corresponding to elements of the second homotopy group $\pi_2(G/J)$, where J is a certain subgroup of G obtained by fixing the boundary conditions. In the simplest example when the gauge group $G = SU(2)$ and $J = U(1)$ $\pi_2(G/J) \cong \pi_2(S^2) \cong \mathbf{Z}$, there is only one integer $N(\omega,\phi) \in \mathbf{Z}$ which classifies the monopole solutions. It is called the *monopole number* or the *topological charge* and is defined by

$$N(\omega,\phi) = \frac{1}{4\pi}\int_{R^3} d^\omega\phi \wedge F \,, \tag{7.8}$$

where we have written F for F_ω. It can be shown that, with suitable decay of $||\phi||$ in R^3 , $N(\omega,\phi)$ is an integer and we have

$$N(\omega,\phi) = \frac{1}{4\pi}\int_{R^3} d^\omega\phi \wedge F = \lim_{r\to\infty}\frac{1}{4\pi}\int_{S_r^2}\langle\phi, F\rangle =$$

$$= deg\{\phi/||\phi|| : S_r^2 \to SU(2)\},$$

where $\langle\ ,\ \rangle$ is the inner product in the Lie algebra. In fact, D. Groisser ([GR5], [GR6]) has proved that in classical $SU(2)$ Yang-Mills-Higgs theories on $\mathbb{R}^3$ with a Higgs field in the adjoint representation, an integer-valued monopole number is canonically defined for any finite action smooth configuration and that the monopole configuration space essentially has the homotopy type of $Maps(S^2, S^2)$ (regarded as maps from the sphere at infinity to the sphere in the Lie algebra $su(2)$) with infinitely many path components, labeled by the monopole number.

The Yang-Mills equations and the Yang-Mills-Higgs equations share several common features. As we noted above locally, the Yang-Mills-Higgs equations are obtained by a dimensional reduction of the pure Yang-Mills equations. Both equations have solutions which are classified by topological invariants. For example, the G-instanton solutions over S^4 are classified by $\pi_3(G)$. For simple Lie groups G this classification goes by the integer defined by the Pontryagin index or the instanton number, whereas the monopole solutions over R^3 are classified by $\pi_2(G/J)$. The first order instanton equations correspond to the first order Bogomolnyi equations and both have solution spaces that are parametrized by manifolds with singularities or moduli spaces. However, there are important global differences in the solutions of the two systems which arise due to different boundary conditions. For example, no translation invariant non-trivial connection over R^4 can extend to S^4 . Extending the analytical foundations laid in [SE1], [UH1], [UH3] Taubes proved the following theorem in [TA2], [TA3].

Theorem 7.2. *There exists a solution to the* $SU(2)$ *Yang-Mills-Higgs equations which is not a solution to the Bogomolnyi equations.* ∎

The corresponding problem regarding the relation of the solutions of the full Yang-Mills equations and those of the instanton equations has recently been solved in [SI4] where the existence of non-instanton solutions to pure Yang-Mills equations over S^4 has been extablished. The basic references for material in this section are H.F. Atiyah, N.J. Hitchin [bAT2], A. Jaffe, C. Taubes [bJA1]. For further developments see [GA4], [GR5], [HI1], [HO1], [DO7], [TA1] .

7.4. Quantization

Quantization of classical fields is an area of fundamental importance in modern mathematical physics. Although there is no satisfactory mathematical theory of quantization, physicists have developed several methods of quantization that can be applied to specific problems. Most successful among these is QED (Quantum Electrodynamics), the theory of quantization of electromagnetic fields. The physical significance of electromagnetic fields is thus well understood at both the classical and the quantum level. Electromagnetic theory is the prototype of classical gauge theories. It is therefore, natural to try to extend the methods of QED to the quantization of gauge theories. We shall restrict our attention to the Feynman path integral method of quantization. Application of this method together with perturbative calculations have yielded some interesting results in the quantization of gauge theories. The starting point of this method is the choice of a Lagrangian defined on the configuration space of classical gauge fields. This Lagrangian is used to define the action functional that enters in the integrand of the Feynman path integral. For example, the Yang-Mills action is given by equation (6.1) and the Yang-Mills-Higgs action is given by equation (7.5). Dimensional reduction allows us to think of a Yang-Mills-Higgs field as a Yang-Mills field on a higher dimensional manifold which is invariant under a certain symmetry group. Thus we may restrict our attention to the Yang-Mills case. In this case the Feynman path integral is given by the following expression.

$$\int_{\mathcal{A}(P)} e^{-\mathcal{A}_{YM}(\omega)} \mathcal{F}(\omega) \mu_\omega,$$

where $\mathcal{F}(\omega)$ is a gauge invariant functional on the configuration space $\mathcal{A}(P)$, and μ_ω is a suitably defined measure on $\mathcal{A}(P)$. This integral is divergent due to the infinite contribution coming from gauge equivalent fields. One way to avoid this difficulty is to observe that the integrand is gauge invariant and hence descends to the orbit space $\mathcal{O} = \mathcal{A}(P)/\mathcal{G}$ and to integrate over this orbit space $\mathcal{O}$. However, the mathematical structure of this space is essentially unknown at this time. Physicists have attempted to get around this difficulty by choosing a section $s : \mathcal{O} \to \mathcal{A}$ and integrating over its image $s(\mathcal{O})$ with a suitable weight factor such as the Fadeev-Popov determinant, which may be thought of as the Jacobian of the change of variables effected by $p_{|s(\mathcal{O})} : s(\mathcal{O}) \to \mathcal{O}$. As we have seen in section 5.2 , this gauge fixing procedure does not work in general, due to the presence of the Gribov ambiguity. Also the Fadeev-Popov determinant is infinite dimensional and needs to be regularized. This is usually done by introducing the anticommuting Grassmann variables called the ghost and antighost fields. The Lagrangian in the action term is then replaced by a new Lagrangian containing these ghost and antighost fields. This new Lagrangian is called the effective Lagrangian. The effective Lagrangian is not gauge invariant, but it is invariant under a special group of transformations involving the ghost and antighost fields. These transformations are called the BRST (Becchi-Rouet-Stora-Tyutin) transformations. On the infinitesimal level the BRST transformations correspond to cohomology operators and define what may be called the BRST cohomology. The non-zero elements of the BRST cohomology are called anomalies in the physics literature. A detailed discussion of the material of this section may be found in the books on quantum field theory referred to in the introduction to this paper. A geometrical interpretation of some of these concepts may be found in [BA2], [CO3], [CO4].

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DICTIONARY OF TERMINOLOGY AND NOTATION

Physics	*Mathematics*
Space-time	Lorentz 4-manifold M
Euclidean space-time	Riemannian 4-manifold M
Gauge group G	structure group of a principal bundle $P(M,G)$ over M
Space of phase factors	total space P of the bundle $P(M,G)$
Global gauge s	a section of the bundle $P(M,G)$
Gauge group bundle	$Ad(P) = P \times_{Ad} G$, where Ad is the adjoint action of G on itself
Gauge transformation	a section of the bundle $Ad(P)$
Gauge algebra bundle	$ad(P) = P \times_{ad} \mathbf{g}$, where ad is the adjoint action of G on $\mathbf{g}$
Infinitesimal gauge transformation	a section of the bundle $ad(P)$
Gauge potential ω on P	connection 1-form $\omega \in \Lambda^1(P,\mathbf{g})$
Gauge potential A on M	$A = s^*(\omega)$
Local gauge t	a section of the bundle $P(M,G)$, restricted to an open set $U \subset M$.
Local gauge potential A_t	$A_t = t^*(\omega)$
Gauge field Ω on P	curvature $d^\omega \omega = \Omega \in \Lambda^2(P,\mathbf{g})$
Gauge field F_ω on M	the 2-form $F_\omega \in \Lambda^2(M, ad(P))$ associated to Ω
Group of generalized gauge transformations	$Diff_M(P) = \{g \in Diff(P) \mid g$ covers $g_M \in Diff(M)\}$
Group $\mathcal{G}$ of gauge transformations	$\mathcal{G} = Aut(P) \subset Diff_M(P)$, the group of bundle automorphisms of P
Group $\mathcal{G}_0 \subset \mathcal{G}$, of based gauge transformations	$\mathcal{G}_0 = Aut_0(P) \subset Aut(P)$, the group of based bundle automorphisms
Gauge algebra $\mathcal{LG}$	Lie algebra $\Gamma(ad(P)) \cong \mathcal{F}_G(P,\mathbf{g})$
generalized Higgs potential	a section of the associated bundle $E(M,F,\tau,P)$

Higgs potential ϕ	a section of the associated bundle $ad\, P = P \times_{ad} \mathfrak{g}$
Higgs field h_ϕ	$h_\phi = d^\omega \phi$
Bianchi identities for the gauge field F_ω	$d^\omega F_\omega = 0$
Yang-Mills field equations for F_ω	$\delta^\omega F_\omega = 0$
Instanton (self-dual Yang-Mills) equations on 4-manifold M	$*F_\omega = F_\omega$, where $*$ is the Hodge operator
Instanton number	Pontryagin index of the bundle P
BPST instanton	canonical $SU(2)$-connection on the quaternionic Hopf fibration of S^7 over S^4
Dirac monopole	canonical $U(1)$-connection on the Hopf fibration of S^3 over S^2
Inertial frames on a space-time manifold (M, g)	the bundle $O(M,g)$ of orthonormal frames on M
Gravitational potential	Levi-Civita connection λ on $O(M,g)$
Gravitational field	curvature R^λ
source free gravitational field equations	$[R^\lambda, *] = 0$, where $*$ is the Hodge operator